



Induction rules, reflection principles, and provably recursive functions

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Abstract

A well-known result (Leivant, 1983) states that, over basic Kalmar elementary arithmetic EA , the induction schema for Σ_n formulas is equivalent to the uniform reflection principle for Σ_{n-1} formulas ($n \geq 1$). We show that fragments of arithmetic axiomatized by various forms of induction rules admit a precise axiomatization in terms of reflection principles as well. Thus, the closure of EA under the induction rule for Σ_n (or Π_{n+1}) formulas is equivalent to ω times iterated Σ_n reflection principle. Moreover, for $k < \omega$, k times iterated Σ_n reflection principle over EA precisely corresponds to the extension of EA by $\leq k$ nested applications of Σ_n induction rule.

The above relationship holds in greater generality than just stated. In fact, we give general formulas characterizing in terms of iterated reflection principles the extension of any given theory (containing EA) by $\leq k$ nested applications of Σ_n or Π_n induction rules. In particular, the closure of a theory T under just one application of Σ_1 induction rule is equivalent to T together with Σ_1 reflection principle for each finite Π_2 axiomatized subtheory of T .

These results have closely parallel ones in the theory of subrecursive function classes. The rules under study correspond, in a canonical way, to natural closure operators on the classes of provably recursive functions. Thus, Σ_1 induction rule precisely corresponds to the primitive recursive closure operator, and Σ_1 collection rule, introduced below, corresponds to the elementary closure operator.

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1. Introduction

It is well known that first-order theories can be defined, over first-order logic, by sets of axioms as well as by sets of rules. An axiom can be viewed as a particular kind of rule with an empty, or with some fixed, provable premise. Vice versa, for a theory T axiomatized by rules, all theorems of T constitute a trivial axiomatization of T by a set of axioms. So, if one identifies a theory with its set of theorems – a point of view especially supported by the model-theoretic tradition in logic – there is no essential difference between rules and axioms.

This paper is devoted to a detailed proof-theoretic analysis of restricted induction rules in arithmetic. Our main results characterize closures of arithmetical theories containing EA by induction rules in terms of axioms. In contrast with the above observation, we are looking for natural and *informative* axiomatizations, rather than for easy but useless ones.

One difficulty in the way of this project lies in the fact that, in general, the closure of a theory T under a given inference rule R not only depends on R , but on also on T . So, a meaningful characterization of a rule must somehow take into account arbitrary theories T (of a given class). This feature requires a somewhat sharper analysis of induction rules than those existing in the proof-theoretic literature [10, 11, 17, 18], for in all these works the authors dealt with the closures under the induction rules of particular base theories, such as EA or PRA .

Our axiomatizations are formulated in terms of iterated *reflection principles*; see [20, 14]. Very roughly, k times iterated reflection principle of relevant arithmetical complexity happens to be the strongest formula that can be inferred from a given finite theory (of relevant complexity) using $\leq k$ nested applications of the induction rule in question. In this sense, our axiomatizations are canonical. In particular, this also allows for general characterizations of the closures of arbitrary extensions of EA under the restricted induction rules.

Our characterizations are *informative* in the sense that they yield several interesting corollaries concerning finite (non)axiomatizability of theories given by induction rules, give wide sufficient conditions for the equivalence of (closures of theories by) Π_{n+1} and Σ_n induction rules, and allow us to give new proofs of several old results, such as the conservativity results for induction schemata over induction rules, characterizations of provably total recursive functions of theories axiomatized by rules, and others.

The rules studied in this paper correspond to natural closure operators on the classes of provably recursive functions of theories, e.g., Σ_1 induction rule precisely corresponds to the primitive recursive closure operator. We also introduce and study a natural version of Σ_1 *collection rule*, which corresponds to the elementary closure operator. This rule is especially useful for the analysis of theories, whose classes of provably recursive functions are not elementarily closed. The role of reflection principles in connection with the rules is similar to the role of universal functions for subrecursive classes w.r.t. the above-mentioned closure operators.

For further discussion we must fix some terminology and formulate a few background results.

Kalmar elementary arithmetic EA is a theory known in several equivalent formulations. When formulated in the standard language of Peano arithmetic *PA* it has the name $IA_0 + \text{EXP}$ and is axiomatized by restricting, in the standard formulation of *PA*, the schema of induction

$$A(0) \wedge \forall x(A(x) \rightarrow A(x+1)) \rightarrow \forall x A(x) \quad (1)$$

to bounded formulas $A(x)$ and by adding a Π_2 axiom stating that the function 2^x is total. It is well-known that $IA_0 + \text{EXP}$ is a finitely axiomatizable theory [4].

In an alternative formulation, the language of *EA* contains function symbols for all Kalmar elementary functions, and mathematical axioms of *EA* are (1) (open) defining equations for all these functions; and (2) the schema of induction for open formulas. It is known that *EA* admits a purely universal (or quantifier free) axiomatization in this language. The two formulations of *EA* are equivalent in the sense that the second theory can be viewed as a conservative ‘definitional extension’ of the first one.

Let us also mention the fact that there exists a finite, purely universal formulation of *EA* in a language with symbols for *finitely many* elementary functions. This fact is closely related to a well-known theorem, originally due to Rødding, stating that the class of Kalmar elementary functions has a finite basis under composition (see, e.g., [7, 9]). We shall sketch a proof of this useful fact, as well as that of the finite basis theorem, in Section 4.

Our results are invariant w.r.t. the choice of the language of *EA*, but not all of their proofs are. For definiteness (unless the opposite is obvious from the context) we assume that *EA* is formulated in the language of *PA*. It is known that *EA* is strong enough to reasonably formalize syntax, provability, Gödel’s incompleteness theorems and partial truth definitions (see [4]). All theories considered below are assumed to contain *EA*. By an *arithmetical theory* we mean a theory formulated in the language of *EA*. Classes of arithmetical Σ_n and Π_n formulas are defined in the usual way (cf. p. 13, [4]). They are invariant w.r.t. the choice of the language of *EA* (modulo *EA*-provable equivalence).

C. Parsons was probably the first to systematically study fragments of *PA* obtained by restricting various forms of induction to classes of the arithmetic hierarchy. In [10, 11], among other things, he showed that, over *EA*, the induction schema (1) for Σ_n formulas $A(x)$, denoted $\Sigma_n\text{-IA}$, is strictly stronger than the corresponding induction rule for Σ_n formulas, $\Sigma_n\text{-IR}$ ($n \geq 1$):

$$A(0), \quad \forall x(A(x) \rightarrow A(x+1)) \vdash \forall x A(x).$$

Parsons demonstrated that many other natural forms of restricted induction over *EA* are equivalent to one of these two. In particular,

$$EA + \Sigma_n\text{-IA} \equiv EA + \Pi_n\text{-IA} \quad (2)$$

(this theory is also often denoted $I\Sigma_n$) and

$$EA + \Sigma_n\text{-IR} \equiv EA + \Pi_{n+1}\text{-IR}. \quad (3)$$

Here the expression $T \equiv U$ means that the theories T and U are deductively equivalent, i.e., have the same set of theorems.

Despite the two results looking very similar, they are rather different in nature, as the reader familiar with their proofs undoubtedly feels. Equivalence (2) actually holds over any theory T containing EA , and this indicates a really tight relationship between the two axiom schemata. On the other hand, it is well-known that equivalence (3) may cease to be true for some theories stronger than EA . For example,

$$I\Sigma_1 + \Sigma_1\text{-IR} \equiv I\Sigma_1 \not\equiv I\Sigma_1 + \Pi_2\text{-IR},$$

because $I\Sigma_1 + \Pi_2\text{-IR}$ proves the consistency of $I\Sigma_1$, e.g., by our results in Section 3. This shows that, from some sharper point of view, $\Sigma_1\text{-IR}$ and $\Pi_2\text{-IR}$ are substantially *different* rules. In order to accurately formulate this point of view we adopt a few rather general definitions.

Since the rules we deal with in this paper typically apply to any one from an infinite collection of premises, we say that a *rule* is a set of *instances*, that is, expressions of the form

$$\frac{A_1, \dots, A_n}{B},$$

where A_1, \dots, A_n and B are formulas. Derivations using rules are defined in the standard way; $T + R$ denotes the closure of a theory T under a rule R and first-order logic. $[T, R]$ denotes the closure of T under *unnested applications* of R , that is, the theory axiomatized over T by all formulas B such that, for some formulas A_1, \dots, A_n derivable in T , $(A_1, \dots, A_n)/B$ is an instance of R .

Definition 1. Let R_1 and R_2 be rules. R_1 is *reducible* to R_2 (denoted $R_1 \leq R_2$) iff, for every theory T containing EA , $[T, R_1] \subseteq [T, R_2]$. R_1 and R_2 are *congruent* ($R_1 \cong R_2$) iff $R_1 \leq R_2$ and $R_2 \leq R_1$.

Informally, $R_1 \leq R_2$ means that an arbitrary application of R_1 can be modeled over EA by unnested applications of R_2 . Notice that \leq is reflexive and transitive, so that \cong is an equivalence relation. For the purposes of this paper we may safely identify congruent rules.

We say that a rule R is *congruent to a set of axioms* U , iff R is congruent to U considered as a trivial schematic rule (with the empty premise), or equivalently, iff $[T, R] \equiv T + U$ for any theory T extending EA . Notice that rules congruent to axiom schemata have a trivial behaviour in the sense that they cannot be applied fruitfully more than once: nested applications of such rules do not yield new theorems.

Definition 2. R_1 is *derivable* from R_2 (denoted $R_1 \leq R_2$), iff for every theory T containing EA , $T + R_1 \subseteq T + R_2$.

In other words, $R_1 \leq R_2$ iff for any application $(A_1, \dots, A_n)/B$ of R_1 there exists a derivation of B from A_1, \dots, A_n using EA and rule R_2 . Clearly, $R_1 \leq R_2$ implies $R_1 \leq R_2$ but not necessarily vice versa. Below we shall see that equivalences of rules established by purely elementary methods can usually be strengthened to congruences. On the other hand, equivalence proofs involving more sophisticated methods usually depend on the choice of a particular base theory and therefore do not yield reducibilities either in the sense of Definition 1 or 2.

Example. We have seen that $\Pi_2\text{-IR} \not\leq \Sigma_1\text{-IR}$, although the closure of EA under each of these rules is the same. On the other hand, obviously $\Sigma_1\text{-IR} \leq \Pi_2\text{-IR}$. Corollary 2.2 in Section 2 shows that $\Pi_n\text{-IR} \leq \Sigma_n\text{-IR}$, for $n \geq 1$, but not vice versa.

The plan of the paper is as follows. In Section 2 we classify various forms of induction rules modulo congruence relation. We shall show that these rules, most commonly, fall into one of the three distinct categories: (a) rules congruent to induction axiom *schemata*; (b) rules congruent to Σ_n induction rule $\Sigma_n\text{-IR}$; (c) rules congruent to Π_n induction rule $\Pi_n\text{-IR}$. (An interesting candidate for falling out of this classification is the induction rule for boolean combinations of Σ_n formulas, which is derivable from, but possibly not reducible to, $\Sigma_n\text{-IR}$ for $n > 1$; see Section 10.)

The question of the axiomatizability of rules of category (a) is trivially settled. So, in the remaining part of the paper we analyze the other two cases. In Section 3 we introduce reflection principles and characterize $\Pi_n\text{-IR}$ for $n \geq 1$. A similar characterization of $\Sigma_n\text{-IR}$ is more difficult and is given in Section 7 for $\Sigma_1\text{-IR}$, and in Section 9 for $\Sigma_n\text{-IR}$, $n > 1$. The characterization of $\Sigma_1\text{-IR}$ requires a rather careful analysis of provably recursive functions of theories axiomatized by this rule. In Section 4 we recall basic facts about provably recursive functions and formulate an easy characterization of these functions for closures of Π_2 axiomatized theories by $\Sigma_1\text{-IR}$. In Section 5 we analyze the question, when the class of provably recursive functions of a theory is elementarily closed. A natural sufficient condition is formulated in terms of Σ_1 collection rule. In Section 6, on the basis of these results, we construct a suitable universal function for the class of provably recursive functions of a finite Π_2 axiomatized extension of EA using only unnested applications of $\Sigma_1\text{-IR}$ over that theory. This allows us to obtain in Section 7 the required characterization of $\Sigma_1\text{-IR}$, and subsequently relativize it to $\Sigma_n\text{-IR}$ for $n > 1$.

It should be said that in the proof of our main results we did not try to be overly laconic. We have included proofs of several results which were formally never used in the main proofs, like a theorem of R. Peter on nested recursion, or the results the use of which could be avoided, like the finite basis theorem for Kalmar elementary functions. It seems to us that proofs of these easy facts (modulo the rest of our techniques) would enhance the reader's general understanding of peculiar phenomena treated in this paper, so we decided to include them. The results of Section 3 of this paper have been earlier announced in [1].

2. Basic equivalences

C. Parsons showed that many natural forms of induction (of restricted arithmetical complexity) over EA are equivalent to either Σ_n -IR or Σ_n -IA. In this section we obtain a few more results of this kind. We classify various forms of induction rules modulo the sharper *congruence* relation. Some of Parsons' equivalences then turn out to be congruences, whereas some others do not. We also examine a few rules that have not been considered by Parsons. In addition to IR we consider the following forms of induction rule:

$$\text{IR}_0 : \forall x (A(x) \rightarrow A(x+1)) \vdash A(0) \rightarrow \forall x A(x)$$

$$\text{IR}_{<} : \forall x (\forall y < x A(y) \rightarrow A(x)) \vdash \forall x A(x)$$

$$\text{LR} : \exists x A(x) \vdash \exists x (A(x) \wedge \forall y < x \neg A(y))$$

As usual, for Γ a class of arithmetical formulas, Γ -IR₀, Γ -IR_<, and Γ -LR will denote the above rules with the restriction that $A \in \Gamma$. We also assume that formulas $A(x)$ may contain free parameters other than x . Everywhere below, whenever we talk about Σ_n or Π_n induction rules or axioms, it will be implicitly assumed that $n \geq 1$.

Proposition 2.1. Σ_n -IR₀ \cong Π_n -IR₀ \cong Σ_n -IR.

Proof. (1) The congruence Σ_n -IR₀ \cong Π_n -IR₀ is proved in analogy with the proof of the equivalence of Σ_n -IA and Π_n -IA (cf. [11]). For example, to show that Σ_n -IR₀ \leq Π_n -IR₀, consider a formula $A(x) \in \Sigma_n$ such that

$$T \vdash \forall x (A(x) \rightarrow A(x+1)).$$

Then for $B(a, x) := \neg A(a \dot{-} x)$ one has

$$T \vdash \forall x (B(a, x) \rightarrow B(a, x+1)),$$

whence

$$[T, \Pi_n\text{-IR}_0] \vdash B(a, 0) \rightarrow \forall x B(a, x)$$

$$\vdash B(a, 0) \rightarrow B(a, a)$$

$$\vdash A(0) \rightarrow A(a).$$

Notice that a similar trick does not work with the rule IR.

(2) Obviously, Σ_n -IR \leq Σ_n -IR₀, so we only have to show that Σ_n -IR₀ \leq Σ_n -IR. Let $\exists y A(y, x) \in \Sigma_n$ with $A(y, x) \in \Pi_{n-1}$, and let

$$T \vdash \forall x (\exists y A(y, x) \rightarrow \exists y A(y, x+1)).$$

Then we have

$$T \vdash \forall x (\exists y (A(a, 0) \rightarrow A(y, x)) \rightarrow \exists y (A(a, 0) \rightarrow A(y, x+1)))$$

and obviously

$$T \vdash \exists y (A(a, 0) \rightarrow A(y, 0)).$$

It follows that

$$\begin{aligned} [T, \Sigma_n\text{-IR}] &\vdash \forall x \exists y (A(a, 0) \rightarrow A(y, x)) \\ &\vdash \exists u A(u, 0) \rightarrow \forall x \exists y A(y, x). \quad \square \end{aligned}$$

Corollary 2.2. $\Pi_n\text{-IR} \leq \Sigma_n\text{-IR}$, $\Sigma_n\text{-IR} \not\leq \Pi_n\text{-IR}$.

Proof. First, $\Pi_n\text{-IR} \leq \Pi_n\text{-IR}_0$, and by Proposition 2.1 $\Pi_n\text{-IR}_0 \leq \Sigma_n\text{-IR}$.

Second, it is easily seen (and was noticed by Parsons) that $EA + \Sigma_{k+1}\text{-IR}$ contains $I\Sigma_k$. On the other hand, by a theorem of Leivant [6] on the optimal complexity of axiomatization of induction, $EA + \Pi_{k+1}\text{-IR}$, being an extension of EA by a set of true Π_{k+1} sentences, cannot contain $I\Sigma_k$. This shows our claim for $n > 1$. For $n = 1$ we notice that, e.g., by Theorem 2 proved in Section 7, $[EA, \Sigma_1\text{-IR}]$ contains the uniform Σ_1 reflection principle for EA (this fact can also be inferred from some results in [21]). This means that $[EA, \Sigma_1\text{-IR}]$ is not contained in any consistent set of Π_1 sentences over EA , in particular, not in $EA + \Pi_1\text{-IR}$. \square

Proposition 2.3. $\Pi_n\text{-IR}_< \cong \Pi_n\text{-IR}$, $\Sigma_n\text{-IR}_< \cong \Sigma_n\text{-IR}$.

Proof. The only nontrivial reduction is $\Sigma_n\text{-IR}_< \leq \Sigma_n\text{-IR}$. (Notice that, if $A(x) \in \Sigma_n$, the formula $\forall y \leq x A(y)$ need not be equivalent to a Σ_n -formula in absence of Σ_n -collection principle, and so the obvious argument does not work.)

Suppose

$$T \vdash \forall x (\forall y < x A(y) \rightarrow A(x)), \quad (4)$$

where $A(x) := \exists u A_0(x, u)$, $A_0(x, u) \in \Pi_{n-1}$. Define

$$B(x) := \exists z \forall y \leq x A_0(y, (z)_y).$$

Here $(z)_y$ denotes the y th element of a sequence coded by z , the standard coding function being Kalmar elementary. Clearly, $B(x) \in \Sigma_n$, and from (4) one readily obtains

$$T \vdash B(0) \wedge \forall x (B(x) \rightarrow B(x+1)).$$

Applying $\Sigma_n\text{-IR}$ once, we get $\forall x B(x)$ and $\forall x \exists y A_0(x, u)$. \square

Now we examine some rules congruent to axiom schemata. The effect of such rules over a theory T is precisely that of adding to T a fixed amount of axioms (that do not depend on T). This idea is spelled out in the following definition.

Definition 3. A rule R is *congruent* to a set of formulas U (denoted $R \cong U$) iff, for every theory T containing EA , $[T, R] \equiv T + U$.

It is not difficult to see that, if $R \cong U$, then we have

$$[[T, R], R] \equiv [T, R] + U \equiv (T + U) + U \equiv T + U \equiv [T, R],$$

and so, such a rule can nontrivially be applied only once. Also notice that in order to demonstrate $R \cong U$ it is enough to check that $[EA, R]$ contains U and that $T + U$ is closed under R for every theory T .

Of the rules congruent to axiom schemes the most obvious one is the usual Gentzen-style rule of induction, which can also be called ‘the induction rule with side formulas’. In Hilbert-style formulation it may look, e.g., as follows:

$$\frac{B \rightarrow \forall x (A(x) \rightarrow A(x+1))}{B \rightarrow (A(0) \rightarrow \forall x A(x))}.$$

It is well-known that, whenever the complexity of the formula A is restricted to, say, Σ_n , this rule provides an alternative axiomatization of $I\Sigma_n$ (over EA). Moreover, the reader may easily check that to derive an instance of Σ_n -IA only one application of the rule is necessary. On the other hand, the fact that $T + \Sigma_n$ -IA is closed under the induction rule with side formulas is obvious, hence the rule is congruent to Σ_n -IA. Of course, such an effect is only possible because no restriction was imposed on the arithmetical complexity of the ‘side formula’ B . Our further examples are of a somewhat more delicate nature.

Recall that, for a class of arithmetical formulas Γ , $\Delta_0(\Gamma)$ -formulas are those obtained from Γ by means of boolean connectives and bounded quantifiers. Parsons [11] essentially proved the following fact.

Proposition 2.4. $\Delta_0(\Sigma_n)$ -IR $\cong \Sigma_n$ -IA.

Proof. To derive an instance of Σ_n -IA apply IR to the following $\Delta_0(\Sigma_n)$ formula:

$$A(0) \wedge \forall x < a (A(x) \rightarrow A(x+1)) \rightarrow \forall x \leq a A(x), \quad (5)$$

where $A(x) \in \Sigma_n$.

To show that $T + \Sigma_n$ -IA is closed under $\Delta_0(\Sigma_n)$ -IR for each theory T notice that an even stronger fact is known: $I\Sigma_n$ contains $\Delta_0(\Sigma_n)$ -IA (cf. [11] or [4, Lemma 2.14, p. 65]). \square

The above proposition has a somewhat paradoxical consequence that $\Delta_0(\Sigma_1)$ -IR turns out to be actually *stronger* than Π_2 -IR over EA . This looks strange because we all are used to the fact that in the standard model of arithmetic $\Delta_0(\Sigma_1)$ sets are Δ_2 and hence strictly lower in the hierarchy than Π_2 sets. No contradiction in mathematics arises from this because EA is a weak enough theory to think (or rather, not to exclude) that $\Delta_0(\Sigma_1)$ sets can be very complex. In fact, Proposition 2.4 provides a relevant instance of Σ_1 -IA of the form (5) as an example to this effect. Now we are ready to examine the least element rule LR.

Proposition 2.5. $\Pi_n\text{-LR} \cong \Delta_0(\Sigma_n)\text{-LR} \cong \Sigma_n\text{-LR}$.

Proof. (1) The first congruence is proved very similarly to the quoted Lemma 2.14 of [4]. We only sketch the argument.

For a formula $A(\vec{x}) := A(x_1, \dots, x_k)$ let ‘ q is a z -piece of A ’ denote the following formula:

$$'q \text{ codes a function } [0; z]^k \rightarrow \{0, 1\}' \wedge \forall x_1, \dots, x_k \leq z (A(\vec{x}) \leftrightarrow 'q(\vec{x}) = 1').$$

We say that A is *piecewise coded* in a theory T iff

$$T \vdash \forall z \exists q 'q \text{ is a } z\text{-piece of } A'.$$

It is readily seen that the class of formulas piecewise coded in a theory T containing EA is closed under boolean connectives and bounded quantifiers.

Now we show that the theory $[EA, \Pi_n\text{-LR}]$ piecewise encodes all Σ_n -formulas. Indeed, for any such formula $A(\vec{x})$ we obviously have

$$EA \vdash \exists q (q : [0; a]^k \rightarrow \{0, 1\} \wedge \forall \vec{x} \leq a (A(\vec{x}) \rightarrow q(\vec{x}) = 1)),$$

because, e.g., one may take for q the function identically equal to 1. Applying $\Pi_n\text{-LR}$ once we get the minimal such q . It faithfully encodes the a -piece of A because the standard coding of finite functions has the property that functions with smaller values are assigned smaller codes. It follows that all Σ_n , and hence all $\Delta_0(\Sigma_n)$, formulas are piecewise coded in $[EA, \Pi_n\text{-LR}]$.

Now it is easy to derive $\Delta_0(\Sigma_n)\text{-LR}$. Let $EA \vdash \exists x A(x)$, where $A(x) \in \Delta_0(\Sigma_n)$. Then we have

$$[EA, \Pi_n\text{-LR}] \vdash \exists x, q (A(x) \wedge 'q \text{ is a } x\text{-piece of } A').$$

For this q , using only elementary induction we can find the minimal x such that $q(x) = 1$. It coincides with the least x such that $A(x)$ holds since q is the x -piece of A .

(2) To demonstrate the second congruence it is sufficient to show that every Π_n formula is piecewise coded in $[EA, \Sigma_n\text{-LR}]$. Let $\forall u A_0(u, x)$ be such a formula, with $A_0 \in \Sigma_{n-1}$. Following the same idea as before, and taking for q the function identically equal to 1, we obtain

$$EA \vdash \exists q \exists u \forall x \leq a (A_0((u)_x, x) \rightarrow q(x) = 1).$$

Using $\Sigma_n\text{-LR}$ take the least such q (and a corresponding u). In order to see that q is as required reason, for any $x \leq a$, as follows:

- If $\forall z A_0(z, x)$, then $A_0((u)_x, x)$ and hence $q(x) = 1$.
- If $\exists z \neg A_0(z, x)$ and $q(x) = 1$, pick any such z and define a sequence u' and a function q' as follows: $(u')_i = (u)_i$, for $i \neq x$, $(u')_x = z$; and $q'(i) = q(i)$, for $i \neq x$, $q'(x) = 0$. Then q' has a smaller code than q and satisfies

$$\forall i \leq a (A_0((u')_i, i) \rightarrow q'(i) = 1),$$

which contradicts the minimality of q . \square

Proposition 2.6. $\Pi_n\text{-LR} \cong \Sigma_n\text{-LR} \cong \Sigma_n\text{-IA}$.

Proof. It is well-known that $\Sigma_n\text{-IA}$ is equivalent to the least number principle for $\Delta_0(\Sigma_n)$ formulas (cf. [4]), hence $T + \Sigma_n\text{-IA}$ is closed under $\Delta_0(\Sigma_n)\text{-LR}$ for any theory T . Now we derive the least number principle for an arbitrary $\Delta_0(\Sigma_n)$ formula $A(x)$. Obviously,

$$EA \vdash \exists x (A(a) \rightarrow A(x)).$$

Using Proposition 2.5 we conclude that $[EA, \Pi_n\text{-LR}]$ contains

$$\exists x ((A(a) \rightarrow A(x)) \wedge \forall y < x \neg(A(a) \rightarrow A(y))).$$

This formula implies

$$\exists x (A(a) \rightarrow (A(x) \wedge \forall y < x \neg A(y)))$$

and

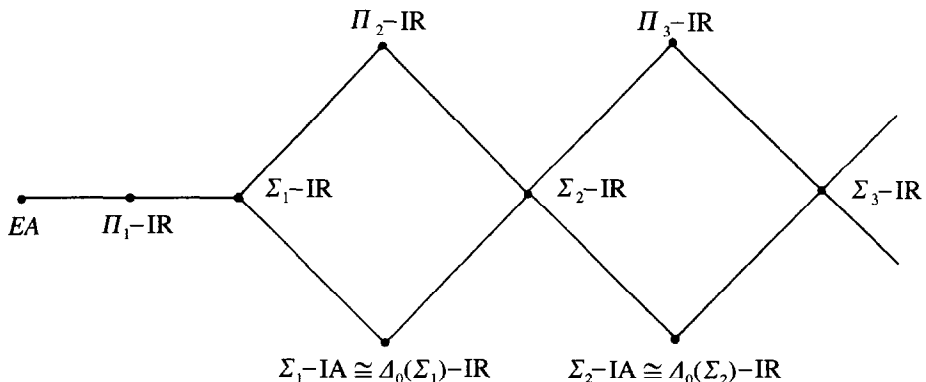
$$\exists z A(z) \rightarrow \exists x (A(x) \wedge \forall y < x \neg A(y)). \quad \square$$

In view of Proposition 2.4 it is natural to ask, what is the strength of the induction rule for boolean combinations of Σ_n formulas, $\mathcal{B}(\Sigma_n)\text{-IR}$. A priori, we can only say that

$$\Sigma_n\text{-IR} \leq \mathcal{B}(\Sigma_n)\text{-IR} \leq \Delta_0(\Sigma_n)\text{-IR},$$

and that at least one of the two inequalities is strict. In the preliminary version of this paper [2] we gave an elementary, although somewhat lengthy, argument showing that $\mathcal{B}(\Sigma_n)\text{-IR}$ is derivable from $\Sigma_n\text{-IR}$. This result can be simplified and somewhat strengthened using more advanced methods. In particular, now we are able to show that $\mathcal{B}(\Sigma_1)\text{-IR}$ and $\Sigma_1\text{-IR}$ are congruent, although it remains open whether this holds for $n > 1$. We shall treat $\mathcal{B}(\Sigma_1)\text{-IR}$ more carefully in Section 10 at the end of the paper.

We summarize the structure of induction rules modulo reducibility (and derivability) relation in the following diagram.



In addition to the already established facts, we remark that neither of the rules Π_{n+1} -IR and Σ_n -IA is derivable from the other, so that all the reducibilities shown on the diagram are proper. Indeed, over EA Σ_n -IA is strictly stronger than Π_{n+1} -IR (see the proof of Corollary 2.2), whereas over $I\Sigma_n$ the latter is stronger than the former, e.g., by Theorem 1 formulated in the next section.

3. Π_n induction rule

In this section we give a characterization of Π_n -IR in terms of iterated reflection principles.

Reflection principles, for an r.e. theory T , are formal schemata expressing the soundness of T , that is, the statement that ‘every sentence provable in T is true’. More precisely, if $\text{Prov}_T(x)$ denotes a canonical Σ_1 provability predicate for T , then the (uniform) reflection principle for T is the schema

$$\forall x (\text{Prov}_T(\ulcorner A(x) \urcorner) \rightarrow A(x))$$

for all formulas $A(x)$. This schema is denoted $\text{RFN}(T)$. *Partial* reflection principles are obtained from it by imposing a restriction that the formula A may only range over a certain subclass Γ of the class of T -formulas. Such schemata will be denoted $\text{RFN}_\Gamma(T)$, and for Γ one usually takes one of the classes Σ_n or Π_n of the arithmetical hierarchy. The following two basic facts on uniform reflection principles are well-known (cf. [20]) and easy:

(1) $\text{RFN}_{\Sigma_n}(T)$ is equivalent to $\text{RFN}_{\Pi_{n+1}}(T)$ over EA , for $n \geq 1$. $\text{RFN}_{\Pi_1}(T)$ is equivalent to $\text{Con}(T)$, the consistency assertion for T .

(2) The schema $\text{RFN}_{\Pi_n}(T)$ is equivalent to a single Π_n sentence (over EA). This follows from the existence of partial truthdefinitions.

An old and well-known result of Kreisel and Lévy [5] says that an alternative axiomatization of Peano Arithmetic over EA can be obtained by replacing the induction schema by the full uniform reflection principle for EA :

$$PA \equiv EA + \text{RFN}(EA).$$

Leivant sharpened this result by showing that the hierarchies of restricted induction schemata and restricted reflection principles over EA actually coincide:

$$I\Sigma_n \equiv EA + \text{RFN}_{\Sigma_{n+1}}(EA).$$

Here we establish a precise relationship between the Π_n induction rule and certain levels of the hierarchy of *iterated* reflection principles.

$IA_0 + \text{SUPEXP}$ is the extension of EA by a Π_2 axiom asserting the totality of superexponentiation function 2_x^y (cf. [4]). A theory T is Π_n *axiomatized*, if all of its nonlogical axioms are Π_n sentences.

Theorem 1. *Let T be an arithmetical theory containing EA . Then, for any $n \geq 2$, $[T, \Pi_n\text{-IR}]$ is equivalent to T together with $\text{RFN}_{\Pi_n}(T_0)$ for all finite Π_{n+1} axiomatized subtheories T_0 of T . This statement also holds for $n=1$, provided T contains $ID_0 + \text{SUPEXP}$.*

Our proof of Theorem 1 is based upon quite standard techniques that combine Tarski's method of partial truthdefinitions with the formalization of the cut-elimination Theorem, and is, in fact, very close to the proof of Leivant's theorem (cf. [6]). The proof admits an easy direct argument, without any use of skolemization. Working in the language of PA , we need a few standard prerequisites.

Sequent calculus: We adopt a variant of the sequent calculus from [16], i.e., sequents are sets of formulas understood as big disjunctions, negations are treated via de Morgan's laws, etc. Unlike in [16], it will be technically convenient for us to only have logical axioms of the form $\Delta, \phi, \neg\phi$, for *atomic* formulas ϕ . It is well-known that the modified calculus is equivalent to the original one, also w.r.t. cut-free provability.

Partial truth definitions: There is a Π_n formula $\text{True}_{\Pi_n}(x)$, which adequately expresses the predicate ' x is a Gödel number of a true Π_n sentence' in EA .² This means that $\text{True}_{\Pi_n}(x)$ is well defined on atomic formulas and provably in EA commutes with boolean connectives and quantifiers, i.e., satisfies Tarski conditions for Π_n formulas. As a result, for any $A(x) \in \Pi_n$, we have

$$EA \vdash \forall x (A(x) \leftrightarrow \text{True}_{\Pi_n}(\ulcorner A(\dot{x}) \urcorner)). \quad (6)$$

For our proof it will be essential that Tarski conditions not only hold locally, for each individual Π_n formula, but also uniformly. In other words, EA proves that, for all $\phi, \psi, \theta, \alpha, \gamma$ such that $\phi, \neg\phi, \psi, \theta, \forall x \gamma(x), \exists x \alpha(x)$ are Π_n sentences,

$$\begin{aligned} \text{True}_{\Pi_n}(\ulcorner \neg\phi \urcorner) &\leftrightarrow \neg \text{True}_{\Pi_n}(\ulcorner \phi \urcorner), \\ \text{True}_{\Pi_n}(\ulcorner \theta \wedge \psi \urcorner) &\leftrightarrow \text{True}_{\Pi_n}(\ulcorner \theta \urcorner) \wedge \text{True}_{\Pi_n}(\ulcorner \psi \urcorner), \\ \text{True}_{\Pi_n}(\ulcorner \theta \vee \psi \urcorner) &\leftrightarrow \text{True}_{\Pi_n}(\ulcorner \theta \urcorner) \vee \text{True}_{\Pi_n}(\ulcorner \psi \urcorner), \\ \text{True}_{\Pi_n}(\ulcorner \exists x \alpha(x) \urcorner) &\leftrightarrow \exists x \text{True}_{\Pi_n}(\ulcorner \alpha(\dot{x}) \urcorner), \\ \text{True}_{\Pi_n}(\ulcorner \forall x \gamma(x) \urcorner) &\leftrightarrow \forall x \text{True}_{\Pi_n}(\ulcorner \gamma(\dot{x}) \urcorner). \end{aligned}$$

Let us stress that ϕ, ψ, \dots here are *variables* over Gödel numbers of sentences, rather than individual sentences. (The standard dots-and-corners notation is somewhat sloppy in this respect. Yet, we hope that this will not create serious problems for the reader.)

On a par with the definition of truth, we also have a reasonable evaluation of terms in EA , that is, a definable Kalmar elementary function $\text{eval}(u, x)$ which provably commutes with $0, ', +, \cdot$ and therefore, for any term $t(x_0, \dots, x_n)$, satisfies

$$EA \vdash \text{eval}(\ulcorner t \urcorner, \langle x_0, \dots, x_n \rangle) = t(x_0, \dots, x_n).$$

² We assume in this section that the class of Π_n formulas contains not only those literally in Π_n form, but also the ones obtained from prenex Π_n formulas using \vee, \wedge , and universal quantification.

Usually, $\text{eval}(u, x)$ is explicitly used in the construction of a truthdefinition for the evaluation of atomic formulas. This implies that the truthdefinition and the evaluation of terms agree in the sense that EA proves that for all Π_n formulas $\phi(x_0, \dots, x_m)$ and terms $t_0(x), \dots, t_m(x)$,

$$\bigwedge_{i=0}^m \text{eval}(\ulcorner t_i \urcorner, \langle x \rangle) = y_i \rightarrow (\text{True}_{\Pi_n}(\ulcorner \phi(t_0(x), \dots, t_m(x)) \urcorner) \leftrightarrow \text{True}_{\Pi_n}(\ulcorner \phi(y_0, \dots, y_m) \urcorner)), \quad (7)$$

and similarly for terms t_i in more than one free variable. We refer the reader to [4] for an elaboration of all the above claims.

Proof of Theorem 1. $[T, \Pi_n\text{-IR}]$ is the theory axiomatized over T by all formulas $\forall x I(x)$ such that $I(x) \in \Pi_n$ and T proves

$$I(0) \wedge \forall x (I(x) \rightarrow I(x+1)). \quad (8)$$

Therefore, first we must show that, for any such $I(x)$, there is a finite Π_{n+1} axiomatized subtheory $T_0 \subseteq T$ such that

$$T + \text{RFN}_{\Pi_n}(T_0) \vdash \forall x I(x).$$

For the axioms of T_0 we simply take the Π_{n+1} formula (8) together with all axioms of EA . Obviously, for every n we have $T_0 \vdash I(\bar{n})$. Furthermore, formalizing this fact in EA we obtain

$$EA \vdash \forall x \text{Prov}_{T_0}(\ulcorner I(x) \urcorner).$$

This implies $\forall x I(x)$ by T_0 -reflection.

Now we must show that

$$[T, \Pi_n\text{-IR}] \vdash \text{RFN}_{\Pi_n}(T_0)$$

for any finite Π_{n+1} axiomatized subtheory $T_0 \subseteq T$. Without loss of generality, we may assume that T itself is a finite Π_{n+1} axiomatized extension of EA . Furthermore, we may assume that the single nonlogical axiom of T has the form $\forall x_0 \dots \forall x_m \neg \alpha(x_0, \dots, x_m)$, where α is a Π_n formula. In particular, this formula accumulates all (finitely many) equality axioms in our language and a finite Π_2 axiomatization of EA .

Consider a cut-free derivation of a sequent of the form $\exists x_0 \dots \exists x_n \alpha, \Pi$, where Π is a set of Π_n formulas. By the subformula property, any formula occurring in this derivation either (a) has the form $\exists x_k \dots \exists x_m \alpha(t_0, \dots, t_{k-1}, x_k, \dots, x_m)$, for some $0 \leq k \leq m$ and terms t_0, \dots, t_{k-1} , or (b) is a Π_n formula.

Now let $I_T(m)$ be a Π_n formula naturally expressing the following:

‘For all p , if p is a cut-free derivation of a sequent of the form $\Gamma, \Pi(a)$, where Γ is a set formulas of type (a) above, $\Pi(a)$ is a set of Π_n formulas, where a stands for all the free variables in Π , and if the height of p is $< m$, then $\forall x \text{True}_{\Pi_n}(\ulcorner \bigvee \Pi(x) \urcorner)$.’

Lemma 3.1. $T \vdash I_T(0) \wedge \forall m (I_T(m) \rightarrow I_T(m+1))$.

Proof. We reason informally within T . $I_T(0)$ trivially holds. We show that $I_T(m)$ implies $I_T(m+1)$. Thus, we are given a cut-free derivation of height $m+1$, of a sequent of the form Γ, Π , where Γ and Π are as above, and we must show that the disjunction of Π is True, in the sense of True_{Π_n} , under every substitution of numerals for free variables in Π . For the rest of the proof we fix an arbitrary substitution of this kind and treat Π as if it were a set of sentences. We distinguish several cases, according to the form of the last rule applied in the given derivation.

Case 1: The sequent Γ, Π is a logical axiom, that is, it has the form $\Delta, \phi, \neg\phi$ for some atomic ϕ . Since all the formulas of type (a) contain at least one existential quantifier and therefore are neither atomic nor negated atomic, both ϕ and $\neg\phi$ must belong to Π . Tarski commutation conditions then imply that

$$\text{True}_{\Pi_n}(\ulcorner \neg\phi \urcorner) \leftrightarrow \neg \text{True}_{\Pi_n}(\ulcorner \phi \urcorner),$$

so we obtain $\text{True}_{\Pi_n}(\ulcorner \phi \urcorner) \vee \text{True}_{\Pi_n}(\ulcorner \neg\phi \urcorner)$ and hence $\text{True}_{\Pi_n}(\ulcorner \bigvee \Pi \urcorner)$.

Case 2: The sequent Γ, Π is obtained by a rule introducing a boolean connective or a quantifier into a formula from Π . All these rules are treated similarly using the subformula property of cut-free derivations and Tarski commutation conditions for True_{Π_n} . For example, the rule for the universal quantifier has the form

$$\frac{\Gamma, \Pi', \phi(a)}{\Gamma, \Pi', \forall x \phi(x)},$$

where a is not free in Γ, Π' . We must show that the formula $\bigvee \Pi' \vee \forall x \phi(x)$ is True. By the induction hypothesis, since a does not occur free in Π' , we know that, for each x , $\bigvee \Pi' \vee \phi(\bar{x})$ is True. Commuting True_{Π_n} with the small disjunction we conclude that, for each x , either $\bigvee \Pi'$ or $\phi(\bar{x})$ is True. Since Π' , and also $\text{True}_{\Pi_n}(\ulcorner \Pi' \urcorner)$, do not depend on x , it follows that either Π' is True, or for every x , $\phi(\bar{x})$ is True. Commuting True_{Π_n} with the universal quantifier and then backwards with the disjunction we conclude that $\Pi' \vee \forall x \phi(x)$ is True. \square

In the next case we shall be more explicit about parameters.

Case 3: The last rule introduces the existential quantifier in front of α , i.e., our derivation has the form

$$\frac{\Gamma', \alpha(t_0(a), \dots, t_{m-1}(a), t_m(a)), \Pi(a)}{\Gamma', \exists x_m \alpha(t_0(a), \dots, t_{m-1}(a), x_m), \Pi(a)}.$$

A free variable a here stands for all the parameters on which Π and the terms t_i may depend.

So, the induction hypothesis is applicable and implies that, for all x , either the disjunction of $\Pi(\bar{x})$, or $\alpha(t_0(\bar{x}), \dots, t_m(\bar{x}))$ is True. We must, reasoning inside T , refute the second alternative.

Notice that, although, in general, t_i are ‘nonstandard’ terms, α is a fixed ‘standard’ Π_{n+1} formula. Therefore Tarski’s commutation lemma (6) can be applied to α , after evaluating the term t . Thus, by (7) and (6) we obtain

$$\begin{aligned} \text{True}_{\Pi_n}(\ulcorner \alpha(t_0(\dot{x}), \dots, t_m(\dot{x})) \urcorner) \wedge \bigwedge_{i=0}^m \text{eval}(t_i, \langle x \rangle) = y_i \rightarrow \text{True}_{\Pi_n}(\ulcorner \alpha(\dot{y}_0, \dots, \dot{y}_m) \urcorner) \\ \rightarrow \alpha(y_0, \dots, y_m) \end{aligned}$$

Since the evaluation function is provably total in EA , it follows that

$$\text{True}_{\Pi_n}(\ulcorner \alpha(t_0(\dot{x}), \dots, t_m(\dot{x})) \urcorner)$$

implies $\exists y_0 \dots y_m \alpha(y_0, \dots, y_m)$, that is, yields a contradiction in T . Thus, we see that, for any t_0, \dots, t_m and x , the formula $\alpha(t_0(\bar{x}), \dots, t_m(\bar{x}))$ cannot be True, hence the disjunction of $\Pi(\bar{x})$ is True.

Case 4: Γ, Π is obtained by a rule introducing any other existential quantifier into a formula from Γ . Then our claim follows immediately from the induction hypothesis, because the Π part of the premise in this case is the same as that of the conclusion. \square

An immediate corollary of the above lemma is that

$$[T, \Pi_n\text{-IR}] \vdash \forall m I_T(m). \quad (9)$$

Notice that for T containing EA and $n \geq 2$, obviously,

$$[T, \Pi_n\text{-IR}] \vdash \text{SUPEXP}.$$

On the other hand, it is well known (cf. [4]) that $IA_0 + \text{SUPEXP}$ is a strong enough theory to prove the Cut-elimination Theorem for first-order logic. In order to derive $\text{RFN}_{\Pi_n}(T)$ we reason inside $[T, \Pi_n\text{-IR}]$, for every particular Π_n formula $A(x)$, as follows.

Suppose $\text{Prov}_T(\ulcorner A(\dot{x}) \urcorner)$. Then the sequent $\exists x_0 \dots \exists x_m \alpha(x_0, \dots, x_m), A(\bar{x})$ is logically provable. By (formalized) cut-elimination theorem we obtain a cut-free proof of this sequent, and by (9) conclude that $\text{True}_{\Pi_n}(\ulcorner A(\dot{x}) \urcorner)$ holds. Tarski commutation lemma (6) then yields $A(x)$. \square

The rest of this section is devoted to various remarks, corollaries and comments concerning Theorem 1. Let, for a fixed $n \geq 1$, $(T)_k^n$ denote the sequence of theories based on iteration of the Π_n reflection principle over T :

$$(T)_0^n \rightleftharpoons T, \quad (T)_{k+1}^n \rightleftharpoons (T)_k^n + \text{RFN}_{\Pi_n}((T)_k^n), \quad (T)_\omega^n \rightleftharpoons \bigcup_{k \geq 0} (T)_k^n.$$

Similarly, $[T, \Pi_n\text{-IR}]_k$ is defined by repeated application of $\Pi_n\text{-IR}$:

$$[T, \Pi_n\text{-IR}]_0 \rightleftharpoons T, \quad [T, \Pi_n\text{-IR}]_{k+1} \rightleftharpoons [[T, \Pi_n\text{-IR}]_k, \Pi_n\text{-IR}].$$

We obviously have

$$T + \Pi_n\text{-IR} \equiv \bigcup_{k \geq 0} [T, \Pi_n\text{-IR}]_k.$$

Since for r.e. T containing EA the schema $\text{RFN}_{\Pi_n}(T)$ is equivalent to a single Π_n sentence, Theorem 1 can be applied repeatedly and we obtain

Corollary 3.2. *Let T be a finite Π_{n+1} axiomatized theory containing EA (or $IA_0 + \text{SUPEXP}$ for $n = 1$). Then*

$$T + \Pi_n\text{-IR} \equiv (T)_{\omega}^n.$$

Moreover, for all $k \geq 1$, we actually have

$$[T, \Pi_n\text{-IR}]_k \equiv (T)_k^n,$$

that is, k (nested) applications of induction rule precisely correspond to k iterations of reflection principle over T .

Corollary 3.3. *For Π_{n+1} axiomatized theories T containing EA (or $IA_0 + \text{SUPEXP}$ for $n = 1$), the closure of T under Π_n induction rule is a reflexive theory, and hence it is not finitely axiomatizable, unless it is inconsistent. The same holds for any extension of $T + \Pi_n\text{-IR}$ by Σ_n sentences.*

Remark 3.4. Theorem 1 shows that some conservation results for fragments of arithmetic and for iterated reflection principles are mutually interderivable. A well-known theorem due to Parsons, Mints, Takeuti and others states that $I\Sigma_n$ is conservative over $EA + \Pi_{n+1}\text{-IR}$ for Π_{n+1} sentences. This result follows at once from Leivant's equivalent characterization of $\Sigma_n\text{-IA}$ as $\text{RFN}_{\Pi_{n+2}}(EA)$ over EA (cf. [6]) and the characterization of $\Pi_{n+1}\text{-IR}$ in terms of reflection principles in Corollary 3.2. Indeed, by the so-called fine structure theorem of Schmerl (cf. [14]) we know that $\text{RFN}_{\Pi_{n+2}}(EA)$ is a Π_{n+1} conservative extension of $(EA)_{\omega}^{n+1}$, which is equivalent to $EA + \Pi_{n+1}\text{-IR}$ by Corollary 3.2.³ On the other hand, this particular case of Schmerl's theorem obviously follows from Parsons' result, too. The relationship between the 4 mentioned results can be summarized in the following diagram:

$$\begin{array}{ccc} I\Sigma_n & \equiv_{\Pi_{n+1}} & EA + \Pi_{n+1}\text{-IR} \\ ||| & & ||| \\ EA + \text{RFN}_{\Pi_{n+2}}(EA) & \equiv_{\Pi_{n+1}} & (EA)_{\omega}^{n+1} \end{array}$$

The 'horizontal' conservation results are due to Parsons and Schmerl, and the 'vertical' equivalences are Leivant's and ours (Corollary 3.2).

An interesting particular case of Theorem 1 concerns the induction rule for Π_1 formulas. It is well-known that the uniform reflection principle for Π_1 formulas for a

³ Schmerl formulated his result for the hierarchy of (transfinitely iterated) reflection principles over PRA , but it is not difficult to check that his proof essentially works over EA as well.

theory T is equivalent to consistency assertion for T , $\text{Con}(T)$. So, Corollary 3.2 can be reformulated as follows.

Corollary 3.5. *For finite Π_2 axiomatized theories T containing $IA_0 + \text{SUPEXP}$,*

$$T + \Pi_1\text{-IR} \equiv T + \text{Con}(T) + \text{Con}(\text{Con}(T)) + \dots \quad (10)$$

Clearly, for a sound theory T , $T + \Pi_1\text{-IR}$ is an extension of T by true Π_1 axioms, and hence both T and $T + \Pi_1\text{-IR}$ have the same class of provably recursive functions. Despite that, $T + \Pi_1\text{-IR}$ is stronger than T and the equivalence (10) gives us a precise measure of its relative strength.

Remark 3.6. In paper [18] there is a confusion concerning $\Pi_1\text{-IR}$. Theorem 2.1.3 of that paper is false for it implies that $EA + \Pi_1\text{-IR}$ contains more provably recursive functions than EA .⁴

Theorem 2.1.7 of that paper states that the closure of EA under k applications of $\Pi_2\text{-IR}$ (in our terminology, $[EA, \Pi_2\text{-IR}]_k$) is conservative over the arithmetic corresponding to the $(k + 3)$ rd class of Grzegorzczuk hierarchy. This theorem is correct and closely parallel to another particular case of our Theorem 2 (cf. Corollary 7.5).

Remark 3.7. A characterization of $\Pi_1\text{-IR}$ for theories weaker than $IA_0 + \text{SUPEXP}$ can be obtained in the spirit of Wilkie and Paris [21]. In this situation the family of consistency assertions w.r.t. proofs of bounded cut-rank $\text{Con}_k(T)$, $k \geq 0$, plays the role of the single consistency assertion $\text{Con}(T)$ for T . Since EA is a strong enough theory to prove Cut-elimination Theorem for derivations of bounded cut-rank, a quick inspection of the given proof of Theorem 1 yields the following result: for T containing EA , $[T, \Pi_1\text{-IR}]$ is equivalent to T together with all $\text{Con}_k(U)$ such that $k \geq 0$ and U is a finite Π_2 axiomatized subtheory of T .

Our next goal is the characterization of Σ_n induction rule in the spirit of Theorem 1. Parsons showed that $\Sigma_n\text{-IR}$ is equivalent to $\Pi_{n+1}\text{-IR}$ over EA . However, the two rules are not congruent and so, a more careful analysis is needed here. Let me explain why the simple proof of Theorem 1 cannot be easily adapted to the Σ_n case.

The technical reason is that the formula $I_T(m)$ in that proof involves a number of outer universal quantifiers, and therefore does not have the required Σ_n form. Some of these quantifiers, e.g., the quantifier over all derivations p , can actually be bounded. One can replace the induction on the height m of a proof by $\text{IR}_<$ over Gödel numbers p of proofs using the fact that, under the standard coding, subderivations of p have smaller Gödel numbers. However, there does not seem to be an easy way to get rid of the quantifier over all substitutions of numerals for free variables in the end-sequent.

⁴ Lemma 2.1.4 is true, but it is not difficult to see that the schema of ‘restricted primitive recursion’ dealt with there is actually equivalent to the unrestricted primitive recursion. So, the proof-theoretic analysis in this lemma, as it is formulated, gives us no more information about the strength of $\Pi_1\text{-IR}$ than the reduction of $\Pi_1\text{-IR}$ to $\Sigma_1\text{-IR}$.

The only possibility here seems to be to keep those variables free, as the parameters of the formula I_T . Yet, this possibility is blocked by the simple fact that some sequents in the proof p may contain many more parameters than the end-sequent, and we ought to take them all into account. There is one rare situation where this difficulty does not arise: simply, if there are no universal quantifiers in the end-sequent. This idea allows us to analyze the Σ_1 induction rule. Then, by skolemization, we will be able to pull the result up in the arithmetical hierarchy. This project is carefully elaborated in the remaining part of the paper.

4. Provably recursive functions

In this section we recall some basic facts about provably (total) recursive functions (p.t.r.fs) of theories and characterize these functions for closures of theories under Σ_1 induction rule. Most of these results are folklore or close to be so.

We shall deal with various classes of number-theoretic functions. The basic class is the class of elementary functions \mathcal{E} . For a class K , $\mathbf{C}(K)$ denotes the closure of $K \cup \mathcal{E}$ under composition. $[K, \mathbf{PR}]$ denotes the closure of $K \cup \mathcal{E}$ under composition and one application of primitive recursion, i.e., the class $\mathbf{C}(F)$, where F is the set of all functions $f(n, a)$ definable by a schema of the form

$$\begin{aligned} f(0, a) &= g(a), \\ f(n+1, a) &= h(f(n, a), n, a), \end{aligned}$$

for $g, h \in \mathbf{C}(K)$. $\mathbf{E}(K)$ is the elementary closure of K , that is, the class of functions obtained from $K \cup \mathcal{E}$ by closure under composition and bounded sums and products. It is well-known (cf. [13]) that, over a sufficiently large stock of initial elementary functions and modulo composition, bounded summation and multiplication are equivalent to bounded recursion, which, in turn, is equivalent to bounded minimization.

Definition 4. A number-theoretic function $f(x)$ is called *provably recursive* in a theory T iff the graph of f can be represented by a Σ_1 formula $\psi(x, y)$ such that

$$T \vdash \forall x \exists! y \psi(x, y).$$

The class of p.t.r.fs of a theory T is denoted $\mathcal{D}(T)$.

It is easy to see that graphs of p.t.r.fs are actually Δ_1 in T . The class $\mathcal{D}(T)$ is closed under composition, but not necessarily elementarily closed, even if T contains EA . This creates for us some additional difficulties, since proof-theoretically it is much more common and pleasant to deal with elementarily closed classes of functions. Sometimes one considers p.t.r.fs *with elementary graphs*, that is, with the formula $\psi(x, y)$

elementary.⁵ These classes of functions are closed under bounded minimization, but not under composition. However, the following obvious proposition holds.

Proposition 4.1. *For a theory T containing EA , every p.t.r.f can be obtained by composition from a p.t.r.f with an elementary graph and a fixed elementary function.*

Proof. Let $\psi(x, y) := \exists z \psi_0(z, x, y)$, where ψ_0 is elementary, define the graph of f , so that

$$T \vdash \forall x \exists! y \psi(x, y).$$

Using the standard pairing function we let

$$\phi(x, y) := \psi_0((y)_0, x, (y)_1) \wedge \forall z < (y)_0 \neg \psi_0(z, x, (y)_1).$$

Then it is not difficult to check that ϕ defines a certain p.t.r.f g in T , ϕ is elementary, and for all n , $f(n) = (g(n))_1$. \square

Since $\mathcal{D}(T)$ only depends on the Π_2 fragment of T , we shall concentrate our attention on Π_2 axiomatized theories.

Definition 5. Let $\pi := \forall x \exists y \phi(x, y) \in \Pi_2$, with ϕ elementary. A function $f(x)$ is called a *witness* of π iff $\forall x \phi(x, f(x))$ holds in the standard model of arithmetic.

Every true Π_2 sentence has a witness. The function $f_\pi(x)$ whose graph is defined by the formula $\phi(x, y) \wedge \forall z < y \neg \phi(x, z)$ is called *the standard witness* of π .

Proposition 4.2. *Let T be a finite Π_2 axiomatized sound extension of EA , and let f be the standard witness of the single axiom of T . Then $\mathcal{D}(T) = \mathbf{C}(f)$.*

Proof. Obviously, f is a p.t.r.f in T , and so $\mathbf{C}(f) \subseteq \mathcal{D}(T)$. The opposite inclusion is, more or less, a direct consequence of Herbrand's Theorem. Consider a purely universal formulation of EA (in a language with symbols for all Kalmar elementary functions), and add to this language a new function symbol f together with the axiom

$$\forall x \phi(x, f(x)),$$

where $\forall x \exists y \phi(x, y)$ is the single axiom of T over EA . Using appropriate Kalmar elementary terms we can get rid of all bounded quantifiers in ϕ . Hence, the resulting theory is a conservative extension of T and has a purely universal axiomatization.

Now suppose $T \vdash \forall x \exists! y \exists z \psi_0(x, y, z)$, where ψ_0 is elementary (and in our formulation also quantifier-free). Since T has a purely universal axiomatization, by Herbrand's theorem we obtain terms $t_1, \dots, t_k, u_1, \dots, u_k$ of the extended language such that

$$T \vdash \psi_0(a, t_1(a), u_1(a)) \vee \dots \vee \psi_0(a, t_k(a), u_k(a)).$$

⁵ Elementary formulas are bounded formulas in the language of EA with symbols for all Kalmar elementary functions.

Clearly, the terms t_i and u_i represent functions in $\mathbf{C}(f)$. Now we let

$$t(x) := \begin{cases} t_1(x) & \text{if } \psi_0(x, t_1(x), u_1(x)), \\ t_2(x) & \text{if } \psi_0(x, t_2(x), u_2(x)) \text{ and } \neg\psi_0(x, t_1(x), u_1(x)), \\ \dots & \dots \\ t_k(x) & \text{if } \psi_0(x, t_k(x), u_k(x)) \text{ and } \neg\psi_0(x, t_i(x), u_i(x)) \text{ for all } i < k, \\ 0 & \text{otherwise.} \end{cases}$$

The function $u(x)$ is defined in a similar manner, with u_i 's in place of t_i 's. Since the function

$$\text{Cond}(x, y, z) := \begin{cases} x & \text{if } z = 0, \\ y & \text{if } z \neq 0 \end{cases}$$

is elementary, the class $\mathbf{C}(f)$ is closed under definitions by cases and so, $t(x)$ and $u(x)$ can be adequately defined by $\mathbf{C}(f)$ terms. For these terms we obviously have $T \vdash \psi_0(a, t(a), u(a))$. It follows that

$$T \vdash \forall x \exists z \psi_0(x, t(x), z),$$

and by the functionality of ψ

$$T \vdash \forall x, y (t(x) = y \leftrightarrow \psi(x, y)).$$

Since all theorems of T are true, ψ represents the graph of $t(x)$ in the standard model. \square

Remark 4.3. We have actually shown that $\mathcal{D}(T) \subseteq \mathbf{C}(f)$ for any witness f of the axiom of T , not just for the standard one.

Corollary 4.4. *Let T be a finite Π_2 axiomatized sound extension of EA . Then the class $\mathcal{D}(T)$ has a finite basis under composition.*

Proof. Follows from the previous proposition and the fact that \mathcal{E} has a finite basis (cf. e.g., [7, 9]). It might be interesting for the reader to notice that, if we had been slightly more careful in the proof of Proposition 4.2, we could actually have inferred the existence of a finite basis in \mathcal{E} from finite axiomatizability of EA .

Consider a finite Π_2 axiomatization of EA in the usual language of arithmetic (see [4]). Introduce finitely many (Kalmar elementary) functions to quantifier-free represent Δ_0 parts of those Π_2 axioms. Then we have to introduce finitely many Skolem functions for these axioms in order to obtain a purely universal conservative extension of EA . Essentially, the same proof as for Proposition 4.2 then shows that every provably recursive function can be defined by a term in the extended language. In the process we would have to introduce a few more elementary functions like $\text{Cond}(x, y, z)$ or pairing functions. We omit the details. \square

Remark 4.5. The converse of the previous corollary does not hold, essentially because of the difference between provably recursive functions and *programs*. For example, the

theory $(EA)_\omega^1$ extends EA purely universally and therefore has the same, finitely based, class of p.t.r.fs. Yet, this theory is not finitely axiomatizable.

Proposition 4.6. *Let T be a finite Π_2 axiomatized sound extension of EA , and let f be the standard witness of the single axiom of T . Then*

$$\mathcal{D}([T, \Sigma_1\text{-IR}]) = [\mathbf{C}(f), \mathbf{PR}].$$

Proof. Let $g(n, x)$ be defined by a schema of primitive recursion

$$\begin{aligned} g(0, x) &= e(x), \\ g(n+1, x) &= h(g(n, x), n, x). \end{aligned}$$

such that $e, h \in \mathbf{C}(f)$. Since all functions in $\mathbf{C}(f)$ are p.t.r.f in T , graphs of e and h are defined by Σ_1 formulas $E(x, y)$ and $H(z, n, x, y) := \exists v H_0(v, z, n, x, y)$, with H_0 elementary.

The graph of g is most naturally defined (in the standard model) by the following formula (that uses elementary coding of sequences):

$$g(n, x) = y \leftrightarrow \exists s \in \text{Seq} ((s)_0 = e(x) \wedge \forall i < n (s)_{i+1} = h((s)_i, i, x) \wedge (s)_n = y).$$

However, in absense of Σ_1 collection principle this formula may not be equivalent to a Σ_1 formula within T . We modify it as follows (a somewhat similar trick was employed earlier in the proof of Proposition 2.3): $g(n, x) = y \leftrightarrow$

$$\exists s, v \in \text{Seq} (E(x, (s)_0) \wedge \forall i < n H_0((v)_i, (s)_i, i, x, (s)_{i+1}) \wedge (s)_n = y). \quad (11)$$

This formula is obviously Σ_1 , and now we shall show the totality of g in $[T, \Sigma_1\text{-IR}]$.

Clearly, $T \vdash \exists y g(0, x) = y$, because $e(x)$ is provably total. In order to see that

$$T \vdash \forall n (\exists y g(n, x) = y \rightarrow \exists y g(n+1, x) = y)$$

we argue informally as follows. Suppose $g(n, x) = y$ and thus we are given two sequences s and v of length $n+1$ satisfying (11). We have to construct appropriate sequences of length $n+2$. Since the function h is provably total, we can find a z such that $h(y, n, x) = z$. Hence there is a w such that $H_0(w, y, n, x, z)$ holds. Pick any such w and add the element z to the end of the sequence s , and w to the end of v . The resulting sequences are as required. Applying $\Sigma_1\text{-IR}$ we obtain

$$[T, \Sigma_1\text{-IR}] \vdash \forall n \exists y g(n, x) = y.$$

To prove the functionality of g we reason as follows. Let $R(n, s, v, x, y)$ denote the elementary part of the formula (11), and suppose we have $R(n, s_1, v_1, x, y_1)$ and $R(n, s_2, v_2, x, y_2)$. We prove $\forall i \leq n (s_1)_i = (s_2)_i$ by induction on i (with n, s_j, v_j, x, y_j as free parameters). Notice that the induction is elementary, although it is applied as a schema rather than as a rule here. Basis and induction step follow at once from the functionality of e and h . So we obtain $(s_1)_n = (s_2)_n$, and therefore $y_1 = y_2$. Notice that the argument for the functionality was actually carried out in T .

Now we shall show that p.t.r.fs of $[T, \Sigma_1\text{-IR}]$ belong to $[C(f), \mathbf{PR}]$. Since $[T, \Sigma_1\text{-IR}]$ is a sound Π_2 axiomatized theory, it suffices to demonstrate that every formula obtained by an application of $\Sigma_1\text{-IR}$ has a witnessing function in the class $[C(f), \mathbf{PR}]$. (Here we actually apply Remark 4.3 rather than Proposition 4.2.)

Consider an arbitrary elementary formula $A(x, y, a)$ such that

$$\begin{aligned} T &\vdash \exists y A(0, y, a), \\ T &\vdash \forall x (\exists y A(x, y, a) \rightarrow \exists y A(x+1, y, a)). \end{aligned}$$

By Proposition 4.2 we obtain functions $e(a)$ and $h(y, x, a)$ in $C(f)$ such that e witnesses $\forall a \exists y A(0, y, a)$, and h witnesses

$$\forall a, x, y \exists z (A(x, y, a) \rightarrow A(x+1, z, a)).$$

Consider a primitive recursion

$$\begin{aligned} g(0, a) &= e(a), \\ g(x+1, a) &= h(g(x, a), x, a). \end{aligned}$$

Straightforward induction on x then shows that $A(x, g(x, a), a)$ holds in the standard model for all x and a . This means that $g(x, a)$ witnesses $\forall x, a \exists y A(x, y, a)$. \square

Corollary 4.7. *For a sound Π_2 axiomatized theory T containing EA ,*

$$\mathcal{D}([T, \Sigma_1\text{-IR}]) = [\mathcal{D}(T), \mathbf{PR}].$$

Proof. We only have to notice that for such theories T , $\mathcal{D}([T, \Sigma_1\text{-IR}])$ is the union of $\mathcal{D}([T_0, \Sigma_1\text{-IR}])$ for all finite subtheories T_0 of T . \square

Remark 4.8. Notice that the requirement of Π_2 axiomatizability of T in the previous corollary cannot, in general, be dropped. Let $T = EA + S$, where S is the sentence $S_1 \rightarrow S_2$ and $S_i = \text{RFN}_{\Pi_{i+1}}(EA)$ for $i = 1, 2$. Clearly, S is a true Π_3 sentence. By Theorem 2 to be proved in Section 7, $[EA, \Sigma_1\text{-IR}] \vdash S_1$; hence $[T, \Sigma_1\text{-IR}] \vdash S_2$ and $\mathcal{D}([T, \Sigma_1\text{-IR}])$ contains all primitive recursive functions. (It is easy to see, cf. e.g. Ono [8], that S_2 implies $I\Sigma_1$ over EA .)

On the other hand, S is Π_2 conservative over EA because for $\pi \in \Pi_2$, $EA + S \vdash \pi$ implies $EA \vdash \neg S_1 \rightarrow \pi$, whence

$$EA \vdash \text{Prov}_{EA}(\ulcorner \pi \urcorner) \wedge \neg \pi \rightarrow \pi,$$

and $EA \vdash \pi$ by Löb's theorem. It follows that $\mathcal{D}(T)$ coincides with \mathcal{E} , and $[\mathcal{D}(T), \mathbf{PR}]$ is properly contained in the class of all primitive recursive functions. \square

5. Elementary closure

As we have noted before, the class $\mathcal{D}(T)$ need not be elementarily closed even if the theory T contains EA . In this section we shall investigate this question in more detail and formulate sufficient conditions for $\mathcal{D}(T)$ to be elementarily closed. A version of the following proposition can be found in [9] with a more complicated proof.

For a function $f(x)$, let $\tilde{f}(n) := \langle f(0), \dots, f(n) \rangle$.

Proposition 5.1. $E(f) = C(\tilde{f})$.

Proof. Obviously $\tilde{f} \in E(f)$, so $C(\tilde{f}) \subseteq E(f)$. For the opposite inclusion we prove that $\sum_{i \leq x} g(i, y) \in C(\tilde{f})$ if $g(x, y) \in C(\tilde{f})$. (Bounded products are treated similarly.)

Let $(z \upharpoonright n)$ denote the initial segment of a sequence z of length $n+1$. This function is clearly Kalmar elementary. Since $g \in C(\tilde{f})$, g can be considered as a term in a language with symbols for all elementary functions and a symbol for \tilde{f} . We systematically replace all occurrences of subterms of the form $\tilde{f}(t)$ in g by $(z \upharpoonright t)$, where z is a new variable. (It does not matter, in what order these occurrences are replaced.) As a result we obtain an elementary function $\tilde{g}(x, y, z)$. Define

$$G(x, y, z) := \sum_{i \leq x} \tilde{g}(i, y, z).$$

We claim that

$$\sum_{i \leq x} g(i, y) = G(x, y, \tilde{f}(b(x, y)))$$

for a certain term $b(x, y) \in C(\tilde{f})$. We only need to ensure that the value of $b(x, y)$ is greater than all values $t(i, y)$ for $i \leq x$, where terms t occur in the context $\tilde{f}(t)$ within g . Notice that \tilde{f} is an increasing function. Therefore, we can majorize each $t(x, y)$ by an increasing function in $C(\tilde{f})$ and take the sum of all these functions as $b(x, y)$. \square

Notice that the previous proposition can be generalized to $E(f_1, \dots, f_n) = C(\tilde{f}_1, \dots, \tilde{f}_n)$ either by encoding f_1, \dots, f_n into a single function using the pairing and projection mechanism, or just by generalizing the proof of Proposition 5.1.

Proposition 5.2. If $f(x)$ is increasing and the graph of f is elementary, then $\tilde{f} \in C(f)$ and therefore $C(f) = E(f)$.

Proof. If f is increasing, for a certain elementary function b we have

$$\tilde{f}(n) = \mu z \leq b(n, f(n)) \quad \forall i \leq n (z)_i = f(i),$$

because the code of a sequence can be estimated elementarily in its length and the largest element ($= f(n)$). \square

Proposition 5.1 also has the following useful corollary.

Proposition 5.3. *For any class of functions K , the class $[K, \mathbf{PR}]$ is elementarily closed.*

Proof. The class $[K, \mathbf{PR}]$ is generated by all functions from K , \mathcal{E} , and functions $f(n, a)$ obtained by primitive recursion

$$f(0, a) = g(a),$$

$$f(n+1, a) = h(f(n, a), n, a)$$

for $g, h \in \mathbf{C}(K)$. By Proposition 5.1 it is sufficient to show that, together with any such f , the class $[K, \mathbf{PR}]$ also contains the function \tilde{f}_1 , where $f_1(n) := f((n)_0, (n)_1)$.

If $(n+1)_0 > 0$, then

$$\begin{aligned} f_1(n+1) &= f((n+1)_0, (n+1)_1) \\ &= h(f((n+1)_0 - 1, (n+1)_1), (n+1)_0 - 1, (n+1)_1) \\ &= h_1(f_1(p(n+1)), n) \end{aligned}$$

for some $h_1 \in \mathbf{C}(K)$, where $p(n) := \langle (n)_0 - 1, (n)_1 \rangle$. Notice that $p(n+1) < n+1$, if $(n+1)_0 > 0$. (The standard pairing function $\langle x, y \rangle$ is monotonic in both arguments.) On the other hand, if $(n+1)_0 = 0$ then, obviously, $f_1(n+1) = g((n+1)_1)$. It follows that \tilde{f}_1 can be defined by the following primitive recursion:

$$\tilde{f}_1(n+1) = \begin{cases} \tilde{f}_1(n) * h_1((\tilde{f}_1(n))_{p(n+1)}, n) & \text{if } (n+1)_0 \neq 0, \\ \tilde{f}_1(n) * g((n+1)_1) & \text{if } (n+1)_0 = 0. \end{cases}$$

Here $*$ denotes the operation of adjoining an element at the end of a sequence. \square

Now we turn to proof-theoretic analogs of the above lemmas.

Definition 6. Let π be a Π_2 sentence. π is *monotonic*, if there is an elementary formula $\phi(x, y)$ such that EA proves that

1. $\pi \leftrightarrow \forall x \exists y \phi(x, y)$,
2. $\phi(x, y) \wedge \phi(x, z) \rightarrow y = z$,
3. $\phi(x_1, y) \wedge \phi(x_2, z) \wedge x_1 \leq x_2 \rightarrow y \leq z$.

Informally, π is monotonic iff it is equivalent to a sentence whose only witness is provably increasing.

Proposition 5.4. *Let T be a Π_2 axiomatized theory containing EA . The following statements are equivalent:*

1. T is axiomatizable over EA by monotonic Π_2 sentences;
2. T is closed under Σ_1 collection rule:

$$\Sigma_1\text{-CR} \quad \forall x \exists y \phi(x, y) \vdash \forall x \exists y \forall u \leq x \exists v \leq y \phi(u, v),$$

where $\phi(x, y) \in \Sigma_1$.

Proof. Clearly, the formula $\forall x \exists y \forall u \leq x \exists v \leq y \phi(u, v)$ implies $\forall x \exists y \phi(x, y)$ in EA and is monotonic, whenever ϕ is elementary. So, we may apply Σ_1 collection rule to all axioms of T and obtain a monotonic axiomatization.

In order to show that Statement 1 implies 2 we take an axiomatization of T over EA by Π_2 formulas whose standard witnesses are monotonic. Then we introduce Skolem functions for all these formulas and replace axioms $\pi := \forall x \exists y \phi(x, y)$ of T by their skolemizations $\forall x \phi(x, f_\pi(x))$. The resulting theory T^* proves monotonicity of all these functions f_π :

$$x_1 \leq x_2 \rightarrow f_\pi(x_1) \leq f_\pi(x_2).$$

Besides, it is conservative over T , and has a purely universal axiomatization (if EA is taken in a universal formulation).

Now assume $T \vdash \forall x \exists y \psi(x, y)$ for a formula $\psi \in \Sigma_1$. By Herbrand's theorem we can obtain a monotonic term $t(x)$ in the extended language such that

$$T^* \vdash \forall x \exists y \leq t(x) \psi(x, y).$$

(This actually is a version of Parikh's theorem for T^* (cf. [4, p. 272]). Here we use the fact that every elementary function can be majorized by an increasing one, and hence any term in the extended language can.) Provable monotonicity of $t(x)$ then implies

$$T^* \vdash \forall x \exists y = t(x) \forall u \leq x \exists v \leq y \psi(u, v).$$

The result follows by conservativity of T^* over T . \square

Corollary 5.5. *A Π_2 sentence $\forall x \exists y \phi(x, y)$ is monotonic iff*

$$EA \vdash \forall x \exists y \phi(x, y) \rightarrow \forall x \exists y \forall u \leq x \exists v \leq y \phi(u, v).$$

Corollary 5.6. *For a sound Π_2 axiomatized theory T containing EA ,*

1. $[T, \Sigma_1\text{-CR}] \equiv T + \Sigma_1\text{-CR}$,
2. $\mathcal{D}(T + \Sigma_1\text{-CR}) = \mathbf{E}(\mathcal{D}(T))$.

Proof. Part 1 follows from the fact that, for a Π_2 axiomatized theory T , $[T, \Sigma_1\text{-CR}]$ can be axiomatized by monotonic Π_2 sentences. The inclusion $\mathcal{D}(T + \Sigma_1\text{-CR}) \supseteq \mathbf{E}(\mathcal{D}(T))$ follows from the fact that $T + \Sigma_1\text{-CR}$ is axiomatizable by a set of monotonic Π_2 sentences, whose witnessing functions are increasing and have elementary graphs, so that the class $\mathcal{D}(T + \Sigma_1\text{-CR})$ is elementarily closed by Proposition 5.2.

By the definition of $\Sigma_1\text{-CR}$, each of the witnessing functions for the axioms of $[T, \Sigma_1\text{-CR}]$ either coincides with one of T , or has the form $\max_{i \leq x} f_\pi(i)$, where π is an axiom of T . Hence, it belongs to $\mathbf{E}(\mathcal{D}(\mathcal{T}))$, and the inclusion $\mathcal{D}([T, \Sigma_1\text{-CR}]) \subseteq \mathbf{E}(\mathcal{D}(T))$ follows by Proposition 4.2. \square

Corollary 5.7. *If a sound theory T containing EA is closed under Σ_1 collection rule, then $\mathcal{D}(T)$ is elementarily closed.*

The following proposition reveals a useful ‘monotonizing’ property of Σ_1 induction rule.

Proposition 5.8. *For any theory T extending EA , $[T, \Sigma_1\text{-IR}]$ is axiomatizable by monotonic Π_2 sentences over T . If T itself is Π_2 axiomatized, $[T, \Sigma_1\text{-IR}]$ is axiomatizable by monotonic Π_2 sentences over EA .*

Proof. The proof is, essentially, a formalization of Proposition 5.3. Suppose $\phi(x, y, a)$ is elementary and

$$\begin{aligned} T &\vdash \exists y \phi(0, y, a), \\ T &\vdash \forall x (\exists y \phi(x, y, a) \rightarrow \exists y \phi(x+1, y, a)). \end{aligned}$$

Then we define

$$\phi'(x, y) := \forall i \leq x \phi((i)_0, (y)_i, (i)_1),$$

and somewhat similarly to the proof of Proposition 5.3 show that

$$\begin{aligned} T &\vdash \exists y \phi'(0, y), \\ T &\vdash \forall x (\exists y \phi'(x, y) \rightarrow \exists y \phi'(x+1, y)). \end{aligned}$$

Applying $\Sigma_1\text{-IR}$ we obtain $\forall x \exists y \phi'(x, y)$ and

$$\forall x \exists y \forall i \leq x \exists v \leq y \phi((i)_0, v, (i)_1).$$

The latter formula is monotonic and implies $\forall a, x \exists y \phi(x, y, a)$. It follows that $[T, \Sigma_1\text{-IR}]$ is axiomatized by monotonic sentences over T .

A similar argument shows that for each theorem of T of the form $\forall x \exists y \phi(x, y)$, with ϕ elementary, the formula $\forall x \exists y \forall u \leq x \exists v \leq y \phi(u, v)$ is provable in $[T, \Sigma_1\text{-IR}]$. So, if T is Π_2 axiomatized, in an axiomatization of $[T, \Sigma_1\text{-IR}]$ the axioms of T can also be replaced by monotonic sentences. \square

Corollary 5.9. *For a sound Π_2 axiomatized theory T containing EA the class $\mathcal{D}([T, \Sigma_1\text{-IR}])$ is elementarily closed.*

Remark 5.10. This fact can also be directly inferred from Proposition 5.3 and Corollary 4.7.

Finally, we formulate a technically very useful proposition that also relies on monotonicity properties of functions and states, roughly, that for a provably increasing function f the induction schema for formulas elementary in f is reducible to the induction schema for formulas elementary in the *graph* of f . This fact is essentially due to Gaifman and Dimitracopoulos [3]. A somewhat weaker version can be found in [4, Proposition 1.3, p. 271] and we follow the idea of these proofs very closely.

Let $\Delta_0(f)$ denote the class of bounded formulas in the language of EA (with symbols for all elementary functions) enriched by a function symbol $f(x)$, where, in particular, f may occur in bounding terms. Let $F(x, y)$ denote the formula $f(x) = y$ defining the graph of f . $\Delta_0(F)$ formulas are those built up from $F(x, y)$ and elementary ones using boolean connectives and quantifiers bounded by elementary functions. Finally, let T be the theory in the above language obtained by adding to all axioms of EA the axiom

$$\forall x, y (x \leq y \rightarrow f(x) \leq f(y))$$

asserting the monotonicity of f .

Proposition 5.11. *Over the theory T the induction schemata for $\Delta_0(f)$ formulas and $\Delta_0(F)$ formulas are deductively equivalent.*

Proof. First of all, notice that any term t in the language of T can be provably majorized by a term provably increasing in each variable (because every elementary function is majorizable by a monotonic one). We fix one such term for every term t and call it \tilde{t} .

Lemma 5.12. *For every term $s(\vec{a})$ there is a monotonic term $t(\vec{a})$ and a $\Delta_0(F)$ formula $\psi(\vec{a}, b, y)$ such that*

$$T \vdash \forall y \geq t(\vec{a}) (s(\vec{a}) = b \leftrightarrow \psi(\vec{a}, b, y)).$$

Proof. The argument goes by induction on the build-up of s . For the induction step one reasons as follows. If $s(\vec{a})$ has the form $g(s_1(\vec{a}))$, where s_1 is a term, and g is either f or an elementary function symbol, then by the induction hypothesis one has a term $t_1(\vec{a})$ and a $\Delta_0(F)$ formula $\psi_1(\vec{a}, b, y)$ such that provably in T

$$s_1(\vec{a}) = b \leftrightarrow \psi_1(\vec{a}, b, y)$$

for $y \geq t_1(\vec{a})$. Then we let

$$t(\vec{a}) := t_1(\vec{a}) + \tilde{s}_1(\vec{a}),$$

and it is easy to see that for $y \geq t(\vec{a})$ there holds

$$g(s_1(\vec{a})) = b \leftrightarrow \exists v \leq y (\psi_1(\vec{a}, v, y) \wedge g(v) = b).$$

A similar reduction applies in the case when the function g has more than one argument. \square

Lemma 5.13. *For every $\Delta_0(f)$ formula $\psi(\vec{a})$ there is a $\Delta_0(F)$ formula $\psi_0(\vec{a}, y_1, \dots, y_n)$ and provably monotonic terms $t_1(\vec{a}), t_2(\vec{a}, y_1), \dots, t_n(\vec{a}, y_1, \dots, y_{n-1})$ such that*

$$T \vdash \bigwedge_{i=1}^n y_i \geq t_i(\vec{a}, y_1, \dots, y_{i-1}) \rightarrow (\psi(\vec{a}) \leftrightarrow \psi_0(\vec{a}, y_1, \dots, y_n)).$$

Proof. We argue by induction on the build-up of the $\Delta_0(f)$ formula ψ . Basis of induction follows from Lemma 5.12, so we concentrate our attention upon the most difficult case, when ψ has the form

$$\forall u \leq s(\vec{a}) \phi(\vec{a}, u). \quad (12)$$

Applying the induction hypothesis to $\phi(\vec{a}, u)$ we obtain a $\Delta_0(F)$ formula $\phi_0(\vec{a}, u, y_1, \dots, y_n)$ and monotonic terms $s_1(\vec{a}, u), \dots, s_n(\vec{a}, u, y_1, \dots, y_{n-1})$ such that, provably in T ,

$$\phi_0(\vec{a}, u, y_1, \dots, y_n) \leftrightarrow \phi(\vec{a}, u), \quad (13)$$

whenever $y_i \geq s_i(\vec{a}, u, y_1, \dots, y_{i-1})$ for all $1 \leq i \leq n$. Besides, by Lemma 5.12 we obtain a monotonic term $r(\vec{a}, u)$ and a $\Delta_0(F)$ formula $\tau(\vec{a}, u, z)$ such that

$$\tau(\vec{a}, u, z) \leftrightarrow u \leq s(\vec{a})$$

for $z \geq r(\vec{a}, u)$.

We introduce two fresh variables, y_{n+1} and y_{n+2} , and let $\psi_0(\vec{a}, y_1, \dots, y_{n+2})$ be defined as follows:

$$\forall u \leq y_{n+1} (\tau(\vec{a}, u, y_{n+2}) \rightarrow \phi_0(\vec{a}, u, y_1, \dots, y_n)). \quad (14)$$

We also let

$$t_i(\vec{a}, y_1, \dots, y_{i-1}) := s_i(\vec{a}, \tilde{s}(\vec{a}), y_1, \dots, y_{i-1})$$

for $i \leq n$, and let $t_{n+1}(\vec{a}, y_1, \dots, y_n) := \tilde{s}(\vec{a})$, and $t_{n+2}(\vec{a}, y_1, \dots, y_{n+1}) := r(\vec{a}, y_{n+1})$.

In order to see that the claim of our lemma holds, that is, that formula (14) is provably equivalent to $\forall u \leq s(\vec{a}) \phi(\vec{a}, u)$ for y_i sufficiently large w.r.t. each other, we first notice that $u \leq y_{n+1}$ implies $r(\vec{a}, u) \leq r(\vec{a}, y_{n+1}) \leq y_{n+2}$ by provable monotonicity of the term r and by the choice of y_{n+2} . It follows that, under these assumptions, $\tau(\vec{a}, u, y_{n+2})$ is equivalent to $u \leq s(\vec{a})$, which implies $u \leq \tilde{s}(\vec{a})$, and by monotonicity of terms s_i for any such u we have

$$\begin{aligned} y_1 &\geq t_1(\vec{a}) = s_1(\vec{a}, \tilde{s}(\vec{a})) \geq s_1(\vec{a}, u), \\ y_2 &\geq t_2(\vec{a}, y_1) = s_2(\vec{a}, \tilde{s}(\vec{a}), y_1) \geq s_2(\vec{a}, u, y_1), \\ &\dots \dots \\ y_n &\geq t_n(\vec{a}, y_1, \dots, y_{n-1}) = s_n(\vec{a}, \tilde{s}(\vec{a}), y_1, \dots, y_{n-1}) \geq s_n(\vec{a}, u, y_1, \dots, y_{n-1}). \end{aligned}$$

It follows that the induction hypothesis is applicable and yields (13). From this it is easy to conclude that formula (12) implies (14). The opposite implication is proved in a similar way. \square

To complete the proof of Proposition 5.11 we prove $\Delta_0(f)$ induction in the form of the least number principle

$$\psi(x, \vec{a}) \rightarrow \exists x' \leq x (\psi(x', \vec{a}) \wedge \forall z < x' \neg \psi(z, \vec{a}))$$

for an arbitrary $\Delta_0(f)$ formula $\psi(x, \vec{a})$. We apply Lemma 5.13 to ψ and reason inside T plus $\Delta_0(F)$ induction as follows.

Assume $\psi(x, \vec{a})$ and that y_1, \dots, y_n satisfy the premise of the implication in Lemma 5.13, so that we may infer $\psi_0(x, \vec{a}, \vec{y})$ from $\psi(x, \vec{a})$ (notice that some such y_1, \dots, y_n provably exist). Applying the least element principle for ψ_0 (variables \vec{y} as well as \vec{a} act as free parameters) we obtain an $x' \leq x$ such that

$$\psi_0(x', \vec{a}, \vec{y}) \wedge \forall z < x' \neg \psi_0(z, \vec{a}, \vec{y}).$$

Now we notice that, by monotonicity of terms t_i , for all $i \leq n$ we have

$$t_i(x', \vec{a}, y_1, \dots, y_{i-1}) \leq t_i(x, \vec{a}, y_1, \dots, y_{i-1}),$$

and so the premise of the implication in Lemma 5.13 is satisfied for x' , as well as for any $z < x'$ (for the same reason). It follows that x' is, indeed, the least number satisfying $\phi(x, \vec{a})$. \square

Remark 5.14. Notice that we have actually reduced $\Delta_0(f)$ induction to the one for $\Delta_0(F)$ formulas whose bounding terms are plain variables.

6. Evaluation

The aim of this section is to show that the universal function for the class of p.t.r.fs of a finite Π_2 axiomatized theory T belongs to $[\mathcal{D}(T), \mathbf{PR}]$, and therefore can be represented in $[T, \Sigma_1\text{-IR}]$. As a by-product we obtain a new and very transparent proof of a theorem of Peter (cf. [12] and also [13]) stating that so-called *nested* recursion on ω is reducible to primitive recursion.

Let $f(x)$ be a function. Every function of the class $\mathbf{C}(f)$ can be represented by a term in a language containing a function symbol for f and finitely many function symbols for a certain basis in \mathcal{E} (cf. Proposition 4.4). We call these functions *initial functions*, and the terms of this language will be called *f-terms*. We fix a natural elementary Gödel numbering of *f-terms*.

The *evaluation function* $\text{eval}_f(e, x)$ for *f-terms* is defined as follows:

$$\text{eval}_f(e, x) := \begin{cases} t((x)_0, \dots, (x)_n) & \text{if } e = \ulcorner t \urcorner \text{ for an } f\text{-term } t(x_0, \dots, x_n), \\ 0 & \text{otherwise.} \end{cases}$$

It will also be technically convenient to unify the two arguments of $\text{eval}_f(e, x)$ and introduce the functions $\text{eval}'_f(x) := \text{eval}_f((x)_0, (x)_1)$ and

$$\overline{\text{eval}}_f(x) := \langle \text{eval}'_f(0), \dots, \text{eval}'_f(x) \rangle.$$

Proposition 6.1. $[\mathbf{C}(f), \mathbf{PR}] = \mathbf{C}(\text{eval}_f) = \mathbf{C}(\overline{\text{eval}}_f)$.

Proof. First we show that both eval_f and $\overline{\text{eval}}_f$ belong to $[C(f), \mathbf{PR}]$. The definition of eval_f can obviously be rewritten as a primitive ‘course of values’ recursion:

1. $\text{eval}_f(e, x) := (x)_i$, if $e = \ulcorner x_i \urcorner$, where x_i is the i th variable,
2. $\text{eval}_f(e, x) := h(\text{eval}_f(\ulcorner t_0 \urcorner, x), \dots, \text{eval}_f(\ulcorner t_m \urcorner, x))$, if $e = \ulcorner h(t_0, \dots, t_m) \urcorner$, and h is an initial function,
3. $\text{eval}_f(e, x) := 0$, if none of the above cases holds.

Since there are only finitely many initial functions, this definition has the form of a definition by cases. The cases are Kalmar elementarily recognizable by the naturality assumption on the coding of f -terms. It is well known and easy to see that the ‘course of values’ recursion defining eval_f can be reduced to the usual primitive recursion for the function $\overline{\text{eval}}_f$, from which eval_f can be recovered as

$$\text{eval}_f(e, x) = (\overline{\text{eval}}_f(\langle e, x \rangle))_{\langle e, x \rangle} \quad (15)$$

(compare with our proof of Proposition 5.3).

Now we shall show that $C(\text{eval}_f)$ contains $[C(f), \mathbf{PR}]$. Consider a primitive recursive definition

$$\begin{aligned} g(0, a) &= g_0(a), \\ g(n+1, a) &= h(g(n, a), n, a) \end{aligned}$$

for some f -terms $g_0(a)$ and $h(x, y, a)$. We shall express $g(n, a)$ in the form

$$\text{eval}_f(s(n), \langle a \rangle)$$

for a function $s(n)$ to be found. Let $\text{num}(n)$ denote the index of a constant f -term with value n , and let $\text{Sub}_{xy}(e, i, j)$ compute the index of an f -term that results in simultaneous substitution of f -terms i and j for variables x and y respectively in an f -term e . It is easy to see that functions Sub and num are elementary. Then we can define $s(n)$ as follows:

$$\begin{aligned} s(0) &:= \ulcorner g_0 \urcorner, \\ s(n+1) &:= \text{Sub}_{xy}(\ulcorner h \urcorner, s(n), \text{num}(n)). \end{aligned} \quad (16)$$

By induction on n one easily shows that $s(n)$ is a Gödel number of an f -term $t_n(a)$ such that $t_n(a) = g(n, a)$ for all a . Hence $\text{eval}_f(s(n), \langle a \rangle) = g(n, a)$ for all a and n . So, it only remains to prove that primitive recursion (16) is bounded. Let $|t|$ denote the length (= number of symbols) of a term with index t . For Sub we have the following estimate:

$$|\text{Sub}_{xy}(e, i, j)| \leq C \cdot |e| \cdot \max(|i|, |j|),$$

because the total number of occurrences of variables x and y in a term e is less than $|e|$. On the other hand, the length of $\text{num}(n)$ is at worst linear in n . So, for large enough n we have

$$|s(n+1)| \leq C_1 \cdot |s(n)|.$$

It follows that $|s(n)|$ grows at most exponentially, and thereby $s(n)$ has a doubly exponential bound. \square

Two immediate consequences of the above proposition are:

Corollary 6.2. *The class $[C(f), \mathbf{PR}]$ is finitely based.*

Corollary 6.3. *The class $C(\text{eval}_f)$ is elementarily closed.*

Another interesting corollary is the reduction of *nested* recursion to primitive recursion. A nested recursive definition may have, e.g., the following form:

$$g(0, a) = g_0(a),$$

$$g(n+1, a) = h_0(g(n, h_1(g(n, a), a)), n, a).$$

In general, one allows arbitrarily deep nestings of g -terms on the right-hand side of the definition, but g must only occur in the context $g(n, \cdot)$, that is, the first argument must always be n . An old result of Peter says that nested recursion is reducible to primitive recursion, and it is relevant for our work as follows.

Suppose we want to evaluate a term $t(u(x))$, where t and u are complex terms. Doing this in the most straightforward manner we must first evaluate u and then t , that is,

$$\text{eval}_f(\ulcorner t(u) \urcorner, x) = \text{eval}_f(\ulcorner t \urcorner, \text{eval}_f(\ulcorner u \urcorner, \langle x \rangle)).$$

We see that eval_f occurs doubly nested on the right-hand side of the equation. The evaluation procedure prescribed by Proposition 6.1 is different: we look at the terms t and u as being decomposed into initial functions, and evaluate only one function at a step. This is a longer process, although it yields the same result.

A natural rule to verify the totality of functions defined by nested recursion is Π_2 induction rule, rather than Σ_1 -IR, which only works for primitive recursive definitions on the face of it.⁶ Therefore, it is not surprising that Peter's theorem is an essential element in Parsons' proof of the equivalence of Π_2 and Σ_1 induction rules. Here we obtain a slightly sharpened version of Peter's result for free.

Corollary 6.4. *The closure of a class K of functions containing \mathcal{E} under one application of nested recursion and composition coincides with $[K, \mathbf{PR}]$.*

Proof. Without loss of generality we may assume that K has the form $C(f)$. Now we almost literally follow the lines of the proof of the second part of Proposition 6.1. A function $g(n, a)$ defined by nested recursion from $C(f)$ can be expressed in the form

⁶ A recently introduced 'Logic of Primitive Recursion' by Sieg and Wainer [19] seems to provide a relevant framework for the analysis of the intensional phenomenon of correspondence between rules and computational schemes.

$\text{eval}_f(s(n), a)$ for a suitable elementary function s . The bound on the rate of growth of s , however, will be slightly worse than before. For sufficiently large n we have

$$|s(n+1)| \leq C \cdot |s(n)|^k,$$

where k is the maximum depth of nestings in the definition of g . However, this means that s grows no faster than triply exponentially. \square

Let T be finite Π_2 axiomatized extension of EA and let f be the standard witness for the single axiom of T . Recall that the graph of f is defined by an elementary formula. We shall show that the evaluation function for f -terms can be naturally represented in $[T, \Sigma_1\text{-IR}]$, and that its basic properties are provable in this theory.

Without loss of generality we may assume that T is formulated in a language containing function symbols for f and for finitely many initial elementary functions. By Propositions 4.6 and 6.1 we know that eval_f is provably recursive in $[T, \Sigma_1\text{-IR}]$, and hence its graph can be represented by a certain Σ_1 formula. This formula can be read off from the primitive recursive definition of eval_f , or rather $\overline{\text{eval}}_f$, using the formalization of primitive recursion (11) in the proof of Proposition 4.6. The following somewhat sharper observation will be essential for us below.

Lemma 6.5. *The graph of the function $\overline{\text{eval}}_f$ is elementary and can be naturally defined by a bounded formula.*

Proof. The formula $\overline{\text{eval}}_f(x) = y$ informally tells that y is a sequence of length $x+1$ such that for all $u \leq x$,

1. If $(u)_0$ is the Gödel number of i th variable, then $(y)_u = ((u)_1)_i$;
2. If $(u)_0$ is the Gödel number of a term of the form $h(t_1, \dots, t_m)$ for an initial function h and for some terms t_1, \dots, t_m (whose Gödel numbers j_1, \dots, j_m are bound to be smaller than $(u)_0$), then $(y)_u = h((y)_{(j_1, (u)_1)}, \dots, (y)_{(j_m, (u)_1)})$;
3. $(y)_u = 0$, otherwise.

Let us stress that Clause 2 can only be stated separately for each individual initial function h . Since the graph of f is elementary, so is the above formula. \square

Lemma 6.6. $[T, \Sigma_1\text{-IR}] \vdash \forall x \exists! y \overline{\text{eval}}_f(x) = y$.

Proof. This is a particular instance of Proposition 4.6. For the definition given in Lemma 6.5, the totality of $\overline{\text{eval}}_f$ can be directly verified using one application of the rule $\Sigma_1\text{-IR}_<$, which is congruent to $\Sigma_1\text{-IR}$. The functionality of $\overline{\text{eval}}_f$ is established within T as in the proof of Proposition 4.6. \square

A corollary of this lemma is that a function symbol for $\overline{\text{eval}}_f$, and therefore the one for eval_f , can be introduced within $[T, \Sigma_1\text{-IR}]$. Since the definitions of $\overline{\text{eval}}_f$ and eval_f are natural, recursive clauses 1–3 from the proof of Proposition 6.1 are provable in $[T, \Sigma_1\text{-IR}]$, and we obtain the following statement.

Lemma 6.7. $[T, \Sigma_1\text{-IR}]$ proves

1. ‘ e codes i th variable’ $\rightarrow \text{eval}_f(e, x) = (x)_i$;
2. $\bigwedge_{i=0}^m$ ‘ e_i codes a term’ $\rightarrow \text{eval}_f(\text{Sub}_{x_0 \dots x_m}(\ulcorner h \urcorner, e_0, \dots, e_m), x) = h(\text{eval}_f(e_0, x), \dots, \text{eval}_f(e_m, x))$, for any initial function $h(x_0, \dots, x_m)$.

The following corollary is standard.

Proposition 6.8. For any f -term $t(x_0, \dots, x_n)$,

$$[T, \Sigma_1\text{-IR}] \vdash \text{eval}_f(\ulcorner t \urcorner, \langle x_0, \dots, x_n \rangle) = t(x_0, \dots, x_n).$$

Proof. By external induction on the build-up of t . \square

To be able to more fruitfully use the inductive clauses for eval_f we need a reasonable amount of induction for formulas involving eval_f .

Proposition 6.9. The theory $[T, \Sigma_1\text{-IR}]$ contains the induction schema for bounded formulas in the language with a function symbol for eval_f .

Proof. Recall that eval_f was defined via the function $\overline{\text{eval}}_f$. We observe two things: (a) the graph of $\overline{\text{eval}}_f$ is elementary, by Lemma 6.5; (b) the function $\overline{\text{eval}}_f$ is provably increasing in $[T, \Sigma_1\text{-IR}]$, for obvious reasons. By Proposition 5.11 $\Delta_0(\overline{\text{eval}}_f)$ induction is reducible to elementary induction, that is, is provable in $[T, \Sigma_1\text{-IR}]$. It remains to notice that $\Delta_0(\text{eval}_f)$ formulas can be translated into $\Delta_0(\overline{\text{eval}}_f)$ formulas using (15). \square

Corollary 6.10. $[T, \Sigma_1\text{-IR}]$ proves that for all terms $t(z)$ in one variable and all terms u ,

$$\text{eval}_f(\text{Sub}_z(\ulcorner t \urcorner, \ulcorner u \urcorner), x) = \text{eval}_f(\ulcorner t \urcorner, \langle \text{eval}_f(\ulcorner u \urcorner, x) \rangle).$$

Proof. By $\Delta_0(\text{eval}_f)$ induction on the build-up of t , with u and x as free parameters. \square

7. Σ_1 induction rule

Theorem 2. Let T be an arithmetical theory containing EA . Then $[T, \Sigma_1\text{-IR}]$ is equivalent to T together with $\text{RFN}_{\Sigma_1}(T_0)$ for all finite Π_2 axiomatized subtheories T_0 of T .

Proof. Exactly as in the proof of Theorem 1 we can show that, if for $I(x) \in \Sigma_1$ the theory T proves

$$I(0) \wedge \forall x (I(x) \rightarrow I(x+1)), \tag{17}$$

then for a suitable finite Π_2 axiomatized subtheory T_0 of T one has

$$T + \text{RFN}_{\Sigma_1}(T_0) \vdash \forall x I(x).$$

(For the axioms of T_0 one may take formula (17) together with all axioms of EA .)

For the opposite inclusion it is sufficient to demonstrate that

$$[T, \Sigma_1\text{-IR}] \vdash \text{RFN}_{\Sigma_1}(T)$$

for finite Π_2 axiomatized theories T . Modulo the work we have done in the previous sections the argument will be similar to the one in [17, Theorems 3.2 and 3.3.]

We introduce a function symbol f for the standard witness for the single axiom of T and finitely many symbols for a suitable basis in \mathcal{E} , so that T attains a purely universal axiomatization. It is also essential that the language of T is finite, and that T has only finitely many nonlogical axioms in the extended language.

We know that $[T, \Sigma_1\text{-IR}]$ has a reasonable evaluation function eval_f for terms in the language of T . Using eval_f first we manufacture a satisfaction predicate for quantifier free formulae of T . The following lemma is well-known and easy.

Lemma 7.1. *To every quantifier free formula $\phi(a)$ we can associate a term $\chi_\phi(a)$ such that*

$$T \vdash \phi(a) \leftrightarrow \chi_\phi(a) = 0. \quad (18)$$

Proof. Notice that, provably in T ,

$$t_1(a) = t_2(a) \leftrightarrow |t_1(a) - t_2(a)| = 0,$$

$$\phi(a) \wedge \psi(a) \leftrightarrow \chi_\phi(a) + \chi_\psi(a) = 0,$$

$$\neg\phi(a) \leftrightarrow 1 - \chi_\phi(a) = 0,$$

whenever the terms χ_ϕ and χ_ψ satisfy equivalence (18) for formulas ϕ and ψ . The statement of the lemma follows by induction on the build-up of ϕ . \square

Obviously, the function

$$\text{trm} : \ulcorner \phi \urcorner \mapsto \ulcorner \chi_\phi \urcorner$$

is Kalmar elementary, and Lemma 7.1 is formalizable in EA . We define

$$\text{Sat}_f(e, a) := (\text{eval}_f(\text{trm}(e), a) = 0).$$

This definition guarantees that Sat_f is faithfully defined on atomic formulas (by Proposition 6.8) and provably commutes with all boolean connectives. For example, provably in $[T, \Sigma_1\text{-IR}]$ we have: for all ϕ, ψ ,

$$\begin{aligned} \text{Sat}_f(\ulcorner \phi \wedge \psi \urcorner, a) &\leftrightarrow \text{eval}_f(\text{trm}(\ulcorner \phi \wedge \psi \urcorner), a) = 0 \\ &\leftrightarrow \text{eval}_f(\ulcorner \chi_\phi + \chi_\psi \urcorner, a) = 0 \\ &\leftrightarrow \text{eval}_f(\text{trm}(\ulcorner \phi \urcorner), a) + \text{eval}_f(\text{trm}(\ulcorner \psi \urcorner), a) = 0 \\ &\leftrightarrow (\text{eval}_f(\text{trm}(\ulcorner \phi \urcorner), a) = 0 \wedge \text{eval}_f(\text{trm}(\ulcorner \psi \urcorner), a) = 0) \\ &\leftrightarrow (\text{Sat}_f(\ulcorner \phi \urcorner, a) \wedge \text{Sat}_f(\ulcorner \psi \urcorner, a)). \end{aligned}$$

So, Tarski commutation conditions are satisfied, and in the usual manner we obtain the following lemma.

Lemma 7.2. *For every quantifier free formula $\phi(x_0, \dots, x_n)$ in the language of T ,*

$$[T, \Sigma_1\text{-IR}] \vdash \text{Sat}_f(\ulcorner \phi \urcorner, \langle x_0, \dots, x_n \rangle) \leftrightarrow \phi(x_0, \dots, x_n).$$

We also notice the following useful property of the function trm that can be seen from our proof of Lemma 7.1: for every open formula $\phi(z_0, \dots, z_n)$ and any terms t_0, \dots, t_n we have

$$\text{trm}(\ulcorner \phi(t_0, \dots, t_n) \urcorner) = \text{Sub}_{z_0 \dots z_n}(\text{trm}(\ulcorner \phi \urcorner), \ulcorner t_0 \urcorner, \dots, \ulcorner t_n \urcorner). \quad (19)$$

This property is formalizable in EA and yields the following fact: $[T, \Sigma_1\text{-IR}]$ proves that for all formulas $\phi(z_0, \dots, z_n)$ and any terms t_0, \dots, t_n ,

$$\text{Sat}_f(\ulcorner \phi(t_0, \dots, t_n) \urcorner, x) \leftrightarrow \text{Sat}_f(\ulcorner \phi \urcorner, \langle \text{eval}_f(\ulcorner t_0 \urcorner, x), \dots, \text{eval}_f(\ulcorner t_n \urcorner, x) \rangle). \quad (20)$$

This essentially follows from (19) and Corollary 6.10.

Now let $\forall x_0 \dots \forall x_m \neg \alpha(x_0, \dots, x_m)$ be the single nonlogical axiom of T (accumulating, in particular, all the equality axioms), with α quantifier free. Consider a cut-free derivation of a sequent of the form $\exists x_0 \dots \exists x_m \alpha(x_0, \dots, x_m), \Delta$, where Δ is a set of quantifier-free formulas. By the subformula property, any formula occurring in this derivation either (a) has the form $\exists x_k \dots \exists x_m \alpha(t_0, \dots, t_{k-1}, x_k, \dots, x_m)$, for some $0 \leq k \leq m$ and terms t_0, \dots, t_{k-1} , or (b) is an open formula. Furthermore, since the rule introducing a universal quantifier is never applied, without loss of generality we may assume that the derivation contains no free variables apart from those of Δ (otherwise, substitute 0 for any such variable everywhere in the proof). Let us call a cut-free derivation satisfying these conditions *normal*.

Lemma 7.3. *The theory $[T, \Sigma_1\text{-IR}]$ proves the uniform reflection principle for quantifier free formulas of T w.r.t. normal provability, that is, the following statement:*

If a sequent of the form Γ, Δ , where Δ consists of open formulas in the language of T and Γ is a set of formulas of type (a) above, has a normal proof, then for all n , $\text{Sat}_f(\ulcorner \bigvee \Delta \urcorner, n)$.

Proof. The argument is similar to the one in the proof of Theorem 1 and, in fact, easier, although there are some subtle formal differences. Reasoning inside $[T, \Sigma_1\text{-IR}]$ we fix an arbitrary substitution of numerals \bar{n} for free variables of Δ everywhere in the given normal derivation and obtain a derivation p of a sequent of the form $\Gamma(\bar{n}), \Delta(\bar{n})$. By the normality, any subderivation q of p has a similar form, and its Gödel number is smaller than p . By induction on the height h of q we prove the following statement:

For all h, q , if q is a subderivation of p of height h and the end sequent of
 (*) q has the form Γ', Δ' , where Γ' is of type (a) and Δ' is quantifier free, then $\text{Sat}_f(\ulcorner \bigvee \Delta' \urcorner, \langle \rangle)$.

Since there are only finitely many subderivations of p , the quantifier over all q in this statement is bounded, and p appears as a free variable. So, the whole induction is an instance of $\Delta_0(\text{eval}_f)$ induction schema, which is available in $[T, \Sigma_1\text{-IR}]$ by Proposition 6.9.

As usual, we consider several cases according to the last rule applied in the subderivation q . The cases of logical axioms and rules of propositional logic are easily treated using commutation properties for Sat_f . The only nontrivial case is that of the existential quantifier in front of α , that is, when the inference has the form

$$\frac{\Gamma'', \alpha(t_0, \dots, t_{m-1}, t_m), \Delta'}{\Gamma'', \exists x_m \alpha(t_0, \dots, t_{m-1}, x_m), \Delta'}.$$

Then by the induction hypothesis and commutation properties for Sat_f we know that either $\text{Sat}_f(\ulcorner \alpha(t_0, \dots, t_{m-1}, t_m) \urcorner, \langle \rangle)$ or $\text{Sat}_f(\ulcorner \bigvee \Delta' \urcorner, \langle \rangle)$ holds. Suppose

$$\text{Sat}_f(\ulcorner \alpha(t_0, \dots, t_m) \urcorner, \langle \rangle),$$

then by (20) we obtain $\text{Sat}_f(\ulcorner \alpha \urcorner, \langle \text{eval}_f(t_0, \langle \rangle), \dots, \text{eval}_f(t_m, \langle \rangle) \rangle)$, whence

$$\alpha(\text{eval}_f(t_0, \langle \rangle), \dots, \text{eval}_f(t_m, \langle \rangle))$$

by Lemma 7.2. This implies $\exists y_0 \dots \exists y_m \alpha(y_0, \dots, y_m)$ and a contradiction in $[T, \Sigma_1\text{-IR}]$.

So, we have demonstrated (*) and, considering the end sequent of the given derivation p , may conclude that $\text{Sat}_f(\ulcorner \bigvee \Delta(\bar{n}) \urcorner, \langle \rangle)$ holds. By (20) this implies $\text{Sat}_f(\ulcorner \bigvee \Delta \urcorner, n)$. \square

Now we are able to complete the proof of Theorem 2. Since $[EA, \Sigma_1\text{-IR}]$ contains SUPEXP and, therefore, proves the Cut-elimination Theorem for first-order logic, it is sufficient to prove the Σ_1 reflection principle for T w.r.t. cut-free provability. We reason inside $[T, \Sigma_1\text{-IR}]$ as follows.

Suppose $\exists x \sigma(x, a)$ is cut-free provable in T , where $\sigma(x, a)$ is quantifier free. Since T is a purely universal theory, by (formalized) Herbrand's Theorem, as in the proof of Proposition 4.2, we can find a f -term $t(a)$ and a normal derivation of the sequent $\exists x_0 \dots \exists x_m \alpha(x_0, \dots, x_m), \sigma(t(a), a)$. By Lemma 7.3 we may conclude that, for all n , $\text{Sat}_f(\ulcorner \sigma(t(a), a) \urcorner, \langle n \rangle)$. Hence, there exists a m such that $\text{Sat}_f(\ulcorner \sigma(x, a) \urcorner, \langle m, n \rangle)$, because for m one can take the value of t , $\text{eval}_f(\ulcorner t \urcorner, \langle n \rangle)$. Lemma 7.2 then yields $\exists y \sigma(y, n)$. \square

Since uniform Π_2 and Σ_1 reflection principles over T are equivalent, we obtain the following important corollary.

Corollary 7.4. *For Π_2 axiomatized theories T containing EA ,*

$$[T, \Sigma_1\text{-IR}] \equiv [T, \Pi_2\text{-IR}].$$

This corollary allows to extend to $\Sigma_1\text{-IR}$ all the facts concerning axiomatizability that we have obtained earlier for Π_2 induction rule. It should be stressed, however,

that these results only apply for Π_2 axiomatized theories, rather than for general Π_3 axiomatized, as in the case of Π_2 -IR.

On the other hand, the transparent analysis of p.t.r.fs of theories axiomatized by Σ_1 -IR allows us to obtain nontrivial results for Π_2 -IR. For example, we have the following result of Sieg for free (cf. [18] and our discussion at the end of Section 3).

Corollary 7.5. *The p.t.r.fs of the theory $[EA, \Pi_2\text{-IR}]_k$ are precisely those of the class \mathcal{E}_{3+k} of the Grzegorczyk hierarchy.*

Proof. This follows from the well-known fact (cf. e.g. [15]) that classes of the Grzegorczyk hierarchy are obtained from \mathcal{E} by iterated application of the operator of primitive recursion, which corresponds to Σ_1 -IR by Corollary 4.7. \square

8. Relativization

Our goal here is to restate Theorem 2 for a language with additional function symbols. Let $\kappa(x)$ be a function. Relativized analogues of classes of functions considered in the proof of Theorem 2 are defined as follows.

$$\mathcal{E}^\kappa := \mathbf{E}(\kappa),$$

$$\mathbf{C}^\kappa(K) := \mathbf{C}(K \cup \mathcal{E}^\kappa).$$

Notice that $\mathbf{C}^\kappa(f) = \mathbf{C}(\bar{\kappa}, f)$, by Proposition 5.1.

Recall that $\Delta_0(\kappa)$ denotes the class of bounded formulas in the language of EA (with symbols for all Kalmar elementary functions) enriched by a function symbol for κ . Σ_1^κ formulas are those of the form $\exists x_1, \dots, x_n A(x_1, \dots, x_n, a)$, where $A \in \Delta_0(\kappa)$. Classes Σ_n^κ and Π_n^κ are defined in a similar manner.

Relativized version of Kalmar elementary arithmetic, EA^κ , is a theory formulated in the language with a function symbol for κ . In addition to the usual axioms of EA it has a schema of induction for $\Delta_0(\kappa)$ formulas. This formulation of EA^κ is not purely universal because of the presence of bounded quantifiers. We show how to reformulate it in a purely universal way.

First of all, we show that one can naturally $\Delta_0(\kappa)$ define the graph of $\bar{\kappa}$ and prove in EA^κ that this relation defines a total function. For example, one can first define an auxiliary function $t(x)$ by

$$t(x) := \mu z \leq x \quad \forall i \leq x \quad \kappa(i) \leq \kappa(z).$$

The graph of t is clearly $\Delta_0(\kappa)$, and since $t(x) \leq x$ holds provably in EA^κ , the totality of t is easily proved by $\Delta_0(\kappa)$ induction. So, we introduce a function symbol for t and then define $m(x) := \kappa(t(x))$. It is easy to see, provably in EA^κ , that

$$m(x) = \max_{i \leq x} \kappa(x).$$

Now we define the graph of $\bar{\kappa}$ as follows:

$$\bar{\kappa}(x) = y : \leftrightarrow y \in \text{Seq} \wedge \text{lh}(y) = x + 1 \wedge \forall i \leq x (y)_i = \kappa(i),$$

where $\text{lh}(y)$ denotes the length of a sequence y . To show that

$$\forall x \exists! y \bar{\kappa}(x) = y \quad (21)$$

we notice that

$$\bar{\kappa}(x) = \mu y. y \in \text{Seq} \wedge \text{lh}(y) = x + 1 \wedge \forall i \leq x \kappa(i) \leq (y)_i.$$

So, given an x we can find a sequence $y = \langle m(x), \dots, m(x) \rangle$ that majorizes κ on the interval $[0, x]$. Then we apply $\Delta_0(\kappa)$ least element principle to find the minimal such y . This proves (21).

The following two useful properties of the function $\bar{\kappa}$ are obviously provable in EA^κ .

1. $\forall x (\bar{\kappa}(x) \in \text{Seq} \wedge \text{lh}(\bar{\kappa}(x)) = x + 1)$,
2. $\forall x, y (x \leq y \rightarrow \bar{\kappa}(x) = \bar{\kappa}(y) \upharpoonright x)$.

In particular, the second property shows that $\bar{\kappa}$ is a provably increasing function. By Proposition 5.11 we know that for such functions $\Delta_0(\bar{\kappa})$ induction is reducible (over EA) to induction for predicates elementary in the graph of $\bar{\kappa}$, i.e., for formulas built up from $\bar{\kappa}(x) = y$ and elementary ones using boolean connectives and quantifiers bounded by elementary functions. Since the graph of $\bar{\kappa}$ is $\Delta_0(\kappa)$, we see that $\Delta_0(\bar{\kappa})$ induction schema is available in EA^κ .

On the other hand, let $EA^{\bar{\kappa}}$ be a theory formulated in the language of EA enriched by a function symbol for $\bar{\kappa}$. Axioms of $EA^{\bar{\kappa}}$ are those of EA plus induction schema for *open* formulas plus formulas 1 and 2 above. We have just seen that it is contained, or rather interpreted, in EA^κ . The opposite containment is also true.

Proposition 8.1. *EA^κ is equivalent to $EA^{\bar{\kappa}}$.*

Proof. First of all, formalizing the proof of Proposition 5.1 we can show that $\mathbf{C}(\bar{\kappa})$ is provably closed under bounded summation.

Lemma 8.2. *For every term $g(x, a)$ in the language of $EA^{\bar{\kappa}}$ we can (effectively) find a term $G(x, a)$ such that $EA^{\bar{\kappa}}$ proves:*

$$G(0, a) = g(0, a),$$

$$G(x + 1, a) = G(x, a) + g(x + 1, a).$$

Notice that any two terms satisfying the above equations are provably equal in $EA^{\bar{\kappa}}$. We shall denote $G(x, a)$ by $\sum_{i \leq x} g(i, a)$. A similar lemma holds for bounded multiplication.

Lemma 8.3. *For every $\Delta_0(\bar{\kappa})$ formula $\phi(a)$ there is a term $\chi_\phi(a)$ such that*

$$EA^{\bar{\kappa}} \vdash \forall x (\phi(x) \leftrightarrow \chi_\phi(x) = 0).$$

Proof. Induction on the build up of ϕ . Boolean connectives are treated as in Lemma 7.1. Bounded quantifiers are translated using Lemma 8.2 as follows:

$$\forall x \leq y \phi(x, a) \leftrightarrow \sum_{x \leq y} \chi_\phi(x, a) = 0, \quad (22)$$

whenever χ_ϕ satisfies the induction hypothesis. We only need to demonstrate equivalence (22) in $EA^{\bar{\kappa}}$ using open induction.

For the implication (\leftarrow) we prove

$$\sum_{x \leq y} \chi_\phi(x, a) = 0 \wedge u \leq y \rightarrow \chi_\phi(u, a) = 0 \quad (23)$$

by an obvious quantifier free induction on y . For the opposite implication (\rightarrow) we reason as follows. Assume $\forall x \leq y \phi(x, a)$. Then prove by quantifier free induction on u , and with y a parameter, that

$$\forall u \leq y \sum_{x \leq u} \chi_\phi(x, a) = 0.$$

Conclude $\sum_{x \leq y} \chi_\phi(x, a) = 0$. Notice that the induction here, being applied as a schema, does not involve the side formula $\forall x \leq y \phi(x, a)$ (which is not quantifier free). \square

From Lemma 8.3 it follows that, using open induction only, we can prove all instances of $\Delta_0(\bar{\kappa})$ induction in $EA^{\bar{\kappa}}$. Now we notice that the function κ can be defined by a term in $EA^{\bar{\kappa}}$:

$$\kappa(x) := (\bar{\kappa}(x))_x.$$

This means that $\Delta_0(\kappa)$ induction is reducible to $\Delta_0(\bar{\kappa})$ induction, and we may conclude that EA^κ is equivalent to $EA^{\bar{\kappa}}$, since the two interpretations we constructed are mutually inverse. \square

Proposition 8.4. *$EA^{\bar{\kappa}}$ has a purely universal axiomatization (in the language with symbols for $\bar{\kappa}$ and for all elementary functions).*

Proof. In the standard axiomatization of $EA^{\bar{\kappa}}$ the instances of quantifier free induction

$$A(0) \wedge \forall y \leq x (A(y) \rightarrow A(y+1)) \rightarrow \forall y \leq x A(y)$$

are bounded, but not literally quantifier free. We show that in an axiomatization of $EA^{\bar{\kappa}}$ these formulas can be replaced by quantifier free ones. To this end we have to improve a little upon Lemma 8.3. We show that in the proof of Lemma 8.3 only a number of purely universal theorems of $EA^{\bar{\kappa}}$ could be used.

Indeed, the treatment of boolean connectives in Lemma 7.1 only requires a finite number of equivalences, like

$$|x - y| = 0 \leftrightarrow x = y$$

or

$$x + y = 0 \leftrightarrow (x = 0 \wedge y = 0).$$

To handle the bounded quantifiers we can simply take open formulas (23) as axioms. However, the proof of the implication

$$\forall x \leq y \chi_\phi(x, a) = 0 \rightarrow \sum_{x \leq y} \chi_\phi(x, a) = 0, \quad (24)$$

poses a problem.

Let $m(y, a)$ be a function defined by

$$m(y, a) := \mu x \leq y. \chi_\phi(x, a) \neq 0.$$

It is well-known that $m(y, a)$ belongs to $\mathcal{E}^{\bar{\kappa}}$ (cf. [13]), and hence, is definable via bounded summation and multiplication. Moreover, in $EA^{\bar{\kappa}}$ one can prove natural properties of μ operator by quantifier free induction, in particular,

$$\sum_{x \leq y} \chi_\phi(x, a) \neq 0 \rightarrow \chi_\phi(m(y, a), a) \neq 0$$

is provable in $EA^{\bar{\kappa}}$. This formula clearly implies (24), and so, we can take it as another open axiom. Thus, we see that Lemma 8.3 follows from a number of purely universal theorems of $EA^{\bar{\kappa}}$. Taking these theorems together with open translations of all instances of quantifier free induction yields an open axiomatization of $EA^{\bar{\kappa}}$. \square

Now we can formulate a relativized version of (a particular case of) Theorem 2.

Theorem 3. *Let T be a finite $\Pi_2^{\bar{\kappa}}$ axiomatized theory. Then*

$$[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}] \vdash \text{RFN}_{\Sigma_1^{\bar{\kappa}}}(T).$$

Proof. We check that everything in the proof of Theorem 2 relativizes. (Notice that the relativized theorem is formulated in such a way that finite axiomatizability of $EA^{\bar{\kappa}}$ is not presumed.) We take a purely universal formulation of $EA^{\bar{\kappa}}$ and introduce a new function symbol f for the standard witness of the (single) $\Pi_2^{\bar{\kappa}}$ axiom of T . At the cost of introducing into the language of T finitely many function symbols for elementary (in $\bar{\kappa}$) functions and adding finitely many purely universal axioms of $EA^{\bar{\kappa}}$, we may assume that the graph of f is open and T has a finite purely universal axiomatization in the language with f . (This follows by compactness from Lemma 8.3 and Proposition 8.4.)

Main steps in the proof of Theorem 2 were as follows: (a) defining the evaluation function for f -terms using only one primitive recursion over $\mathbf{C}(f)$; (b) proving the totality and natural commutation properties for eval_f inside $[T, \Sigma_1\text{-IR}]$; (c) showing that $\Delta_0(\text{eval}_f)$ induction schema is available in $[T, \Sigma_1\text{-IR}]$; and (d) proving uniform reflection principle for open formulas of T (in the language with f) by $\Delta_0(\text{eval}_f)$ induction.

Since $\mathbf{C}^{\bar{\kappa}}(f) = \mathbf{C}(\bar{\kappa}, f)$ and the graph of f is elementary in $\bar{\kappa}$, as in Lemma 6.5 we obtain a natural $\Delta_0(\bar{\kappa})$ definition of the graph of the evaluation function $\overline{\text{eval}}_f^{\bar{\kappa}}(x)$ for terms in the language of T . (This only amounts to adding the function $\bar{\kappa}$ to the list of initial functions.) For this definition one can directly show the totality of $\overline{\text{eval}}_f^{\bar{\kappa}}(x)$ using one application of $\Sigma_1^{\bar{\kappa}}\text{-IR}_<$, which can be reduced to an application of $\Sigma_1^{\bar{\kappa}}\text{-IR}$ by the same proof as in Proposition 2.3. This shows a relativized version of Lemma 6.6:

Lemma 8.5. $[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}] \vdash \forall x \exists! y \overline{\text{eval}}_f^{\bar{\kappa}}(x) = y.$

A corollary is that the function symbols for $\overline{\text{eval}}_f^{\bar{\kappa}}$ and $\text{eval}_f^{\bar{\kappa}}$ can be introduced in $[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}]$. Lemmas 6.7 and 6.8 then remain essentially unchanged, with the understanding that $\bar{\kappa}$ is included in the list of initial functions. Next we obtain a relativized version of Proposition 6.9.

Lemma 8.6. *The theory $[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}]$ contains the induction schema for bounded formulas in the language of $EA^{\bar{\kappa}}$ enriched by a function symbol for $\text{eval}_f^{\bar{\kappa}}$.*

This follows, exactly as in the proof of Proposition 6.9, from the facts that the graph of f is elementary in $\bar{\kappa}$ and that $\overline{\text{eval}}_f^{\bar{\kappa}}$ is provably increasing. Of course, we rely on a relativized version of Proposition 5.11:

Lemma 8.7. *Over $EA^{\bar{\kappa}} + \forall x, y (x \leq y \rightarrow f(x) \leq f(y))$ the induction schemata for $\Delta_0(\bar{\kappa}, f)$ formulas and $\Delta_0(\bar{\kappa}, F)$ formulas are deductively equivalent, where F is the formula $f(x) = y$ representing the graph of f .*

The proof of this proposition goes as before, using the fact that $\bar{\kappa}$ is provably increasing in $EA^{\bar{\kappa}}$, and that the induction schema for $\Delta_0(\bar{\kappa})$ formulas is available in $EA^{\bar{\kappa}}$. (It can also be inferred just as a *corollary* of Proposition 5.11 for a language with the two monotonic function symbols.)

The rest of the proof needs little checking. The evaluation function gives rise to a natural satisfaction predicate in $[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}]$ for quantifier free formulas of T , $\text{Sat}_f^{\bar{\kappa}}(e, x)$. Tarski commutation conditions directly follow from the commutation properties of $\text{eval}_f^{\bar{\kappa}}(e, x)$, as before, and we arrive at a relativized version of Lemma 7.3.

Lemma 8.8. *The theory $[EA^{\bar{\kappa}} + T, \Sigma_1^{\bar{\kappa}}\text{-IR}]$ proves the uniform reflection principle for quantifier free formulas of T w.r.t. (normal) cut-free provability.*

Here we essentially only rely on the fact that T is a finite and purely universal theory, Tarski commutation properties for $\text{Sat}_f^{\bar{\kappa}}(e, x)$, and the availability of $\Delta_0(\text{eval}_f^{\bar{\kappa}})$ induction schema. Theorem 3 follows from this lemma in the usual way. \square

Remark 8.9. Obviously, the analog of Theorem 3 also holds for extensions of the language of arithmetic by more than one additional function symbol $\bar{\kappa}$.

9. Σ_n induction rule

In this section we generalize the results of Section 7 to $\Sigma_n\text{-IR}$ for an arbitrary $n \geq 1$. Our main result is formulated as follows.

Theorem 4. *Let T be an arithmetical theory containing $I\Sigma_n$. Then $[T, \Sigma_{n+1}\text{-IR}]$ is equivalent to T together with $\text{RFN}_{\Sigma_{n+1}}(T_0)$ for all finite Π_{n+2} axiomatized subtheories T_0 of T .*

Corollary 9.1. *For Π_{n+2} axiomatized theories T containing $I\Sigma_n$,*

$$[T, \Sigma_{n+1}\text{-IR}] \equiv [T, \Pi_{n+2}\text{-IR}].$$

The same result holds for $\Sigma_{n+2} \cup \Pi_{n+2}$ axiomatized extensions of $I\Sigma_n$.

Proof. If T_0 is a finite extension of $I\Sigma_n$ axiomatized by a Π_{n+2} sentence π and a Σ_{n+2} sentence σ , then

$$T_0 + \text{RFN}_{\Pi_{n+2}}(T_0) \equiv T_0 + \text{RFN}_{\Pi_{n+2}}(I\Sigma_n + \pi),$$

by formalized deduction theorem, and $\text{RFN}_{\Pi_{n+2}}(I\Sigma_n + \pi)$ is provable in $[I\Sigma_n + \pi, \Sigma_{n+1}\text{-IR}]$. So, $[T, \Sigma_{n+1}\text{-IR}]$ proves $\text{RFN}_{\Pi_{n+2}}(T_0)$ for any finite subtheory T_0 of T , exactly as by Theorem 1 $[T, \Pi_{n+2}\text{-IR}]$ does. \square

We see that the theorem and its corollary only apply to theories T containing $I\Sigma_n$, rather than to arbitrary extensions of EA . This seems to be a fairly restrictive requirement. Recall, however, that $[EA, \Sigma_{n+1}\text{-IR}]$ contains and is, in fact, equivalent to $I\Sigma_n$. It follows that just a single application of $\Sigma_{n+1}\text{-IR}$ brings everything into the class of theories containing $I\Sigma_n$, where Theorem 4 applies. So, we obtain the following corollary.

Corollary 9.2. *For any Π_{n+2} axiomatized extension T of EA , k applications of Π_{n+2} induction rule over T are reducible to $k + 1$ applications of $\Sigma_{n+1}\text{-IR}$:*

$$[T, \Pi_{n+2}\text{-IR}]_k \subseteq [T, \Sigma_{n+1}\text{-IR}]_{k+1}.$$

I do not know if this result is optimal, that is, if $k + 1$ applications of $\Sigma_{n+1}\text{-IR}$ on the right hand side can, in general, be decreased to k applications. However, we have the following result.

Corollary 9.3. *Let T be a Σ_{n+2} axiomatized extension of EA . Then $[T, \Sigma_{n+1}\text{-IR}]$ is equivalent to T together with $\text{RFN}_{\Sigma_{n+1}}(T_0)$ for all finite subtheories T_0 of T . Hence, over such theories, for any k ,*

$$[T, \Pi_{n+2}\text{-IR}]_k \equiv [T, \Sigma_{n+1}\text{-IR}]_k.$$

Proof. Let T_0 be a finite (Σ_{n+2} axiomatized) subtheory of T . First of all, we notice that

$$T_0 + \text{RFN}_{\Sigma_{n+1}}(T_0) \equiv T_0 + \text{RFN}_{\Sigma_{n+1}}(EA),$$

by formalized deduction theorem. We have already noticed before that $[T, \Sigma_{n+1}\text{-IR}]$ proves $\Sigma_n\text{-IA}$, and by Leivant's theorem $I\Sigma_n$ contains $\text{RFN}_{\Sigma_{n+1}}(EA)$. (Alternatively,

this fact can be seen from our proof of Theorem 4 below.) So, $[T, \Sigma_{n+1}\text{-IR}]$ proves $\text{RFN}_{\Sigma_{n+1}}(T_0)$. The opposite inclusion is proved in the usual way.

After the first application of $\Sigma_{n+1}\text{-IR}$ we obtain a theory which is a $\Sigma_{n+2} \cup \Pi_{n+2}$ axiomatized extension of $I\Sigma_n$. So, the second claim of the corollary follows by Corollary 9.1. \square

Now we turn to the proof of Theorem 4. For the sake of clarity of presentation we first give a proof of this theorem for $n = 1$.

Let T be a finite extension of $I\Sigma_1$ having, apart from the axioms of EA , the only Π_3 axiom

$$\tau := \forall u \exists v \forall w \tau_0(u, v, w),$$

where τ_0 is bounded. Let

$$\phi(x) := \exists u \forall v \phi_0(u, v, x)$$

be an arbitrary Σ_2 formula, with ϕ_0 bounded. We are going to show that

$$[T, \Sigma_2\text{-IR}] \vdash \forall x (\text{Prov}_T(\ulcorner \phi(x) \urcorner) \rightarrow \phi(x)).$$

To this end, first we introduce Skolem functions in order to eat up the innermost universal quantifiers in τ and ϕ , i.e., new function symbols $\kappa(x)$ and $v(x)$ together with the following axioms:

$$\forall u \leq x [\exists w \neg \tau_0((u)_0, (u)_1, w) \rightarrow \exists w \leq v(x) \neg \tau_0((u)_0, (u)_1, w)], \quad (25)$$

$$\forall u \leq x [\exists v \neg \phi_0((u)_0, v, (u)_1) \rightarrow \exists v \leq \kappa(x) \neg \phi_0((u)_0, v, (u)_1)]. \quad (26)$$

Let U be a theory obtained by adding to $EA^{\kappa, v}$ axioms (25) and (26). Obviously, U has a $\Pi_1^{\kappa, v}$ axiomatization.

Lemma 9.4. *There is a non-relativizing interpretation $(\cdot)^-$ of U in $I\Sigma_1$ such that*

- (a) $(\cdot)^-$ is identical on formulas in the language of EA ;
- (b) If $A \in \Sigma_1^{\kappa, \bar{v}}$ then $(A)^-$ is equivalent to a Σ_2 formula in $I\Sigma_1$.

Proof. Graph of the function v will be defined by a formula $v(x) = y$ naturally expressing that y is the least z such that

$$\forall u \leq x [\exists w \neg \tau_0((u)_0, (u)_1, w) \rightarrow \exists w \leq z \neg \tau_0((u)_0, (u)_1, w)].$$

Notice that this formula is $\Delta_0(\Sigma_1)$. To show that $I\Sigma_1$ proves $\forall x \exists! y v(x) = y$ we make use of the fact (cf. [4, p. 69]) that $I\Sigma_1$ contains the so-called *strong collection* schema for Σ_1 formulas A :

$$\forall x \exists y \forall u \leq x (\exists z A(u, z, a) \rightarrow \exists z \leq y A(u, z, a)).$$

Then, taking $\neg\tau_0((u)_0, (u)_1, z)$ for A and subsequently applying the Π_1 least number principle to select the (unique) minimal y shows that v is total and functional. The graph of κ is defined similarly.

Now we notice that the functions κ and v thus introduced in $I\Sigma_1$ are monotonic in the sense that $I\Sigma_1$ proves

$$\forall x, y (x \leq y \rightarrow v(x) \leq v(y)). \quad (27)$$

By Proposition 5.11 it follows that the induction schema for $\Delta_0(\kappa, v)$ formulas is reducible over EA plus (27) to the induction schema for bounded formulas in the graphs of v and κ . Since the graphs of v and κ are interpreted as $\Delta_0(\Sigma_1)$ formulas, this means that $I\Sigma_1$ interprets $\Delta_0(\kappa, v)$ induction. It is easy to check that interpretations of the axioms (26) and (25) are provable in $I\Sigma_1$.

Property (a) is part of the definition of the constructed interpretation, and for (b) it is sufficient to demonstrate that $\Delta_0(\bar{\kappa}, \bar{v})$ formulas are Δ_2 in $I\Sigma_1$ under the interpretation in question. By Lemma 8.3 every $\Delta_0(\bar{\kappa}, \bar{v})$ formula is equivalent to an open formula in $EA^{\bar{\kappa}, \bar{v}}$. Such formulas are obviously equivalent in $EA^{\bar{\kappa}, \bar{v}}$ to Δ_1 formulas in the graphs of $\bar{\kappa}$ and \bar{v} . Since the graphs of $\bar{\kappa}$ and \bar{v} are $\Delta_0(\Sigma_1)$, and by Theorem 2.25 (p. 68) of [4] $\Delta_0(\Sigma_1)$ formulas are Δ_2 in $I\Sigma_1$, so are the interpretations of arbitrary $\Delta_0(\bar{\kappa}, \bar{v})$ formulas. \square

Remark 9.5. The results referred to in the proof of the above lemma are all obtained by purely elementary methods.

An obvious corollary of Lemma 9.4 is the fact that U is conservative over $I\Sigma_1$, and this fact can be seen to be provable in EA . (A careful reader may notice that below we only need to interpret a finite fragment of U , and for finite theories such a formalization is immediate.)

Now we observe that the function

$$v'(u, v) := \mu z \leq v(\langle u, v \rangle). \neg\tau_0(u, v, z)$$

is elementary in v , and therefore can be defined by a term in $EA^{\bar{\kappa}, \bar{v}}$. By axiom (25) we may then infer that U proves

$$\forall z \tau_0(u, v, z) \leftrightarrow \tau_0(u, v, v'(u, v)). \quad (28)$$

Let T^+ be a theory in the language of U obtained by adding to U the axiom τ , that is, $T^+ := U + \tau$. By (28), T^+ has a $\Pi_2^{\bar{\kappa}, \bar{v}}$ axiomatization. Besides, since $T \equiv EA + \tau$ contains $I\Sigma_1$, T^+ is a (provably) conservative extension of T . Indeed, for any formula A in the language of T , $T^+ \vdash A$ implies $U \vdash \tau \rightarrow A$, whence $I\Sigma_1 \vdash \tau \rightarrow A$ and $T \vdash A$, by Lemma 9.4.

Reasoning in a similar way, we obtain a term $\kappa'(u, x)$ of $EA^{\bar{\kappa}, \bar{v}}$ such that

$$U \vdash \phi(x) \leftrightarrow \exists u \phi_0(u, \kappa'(u, x), x).$$

Since T^+ is a provably conservative extension of T , this yields

$$EA \vdash \text{Prov}_{T^+}(\ulcorner \exists u \phi_0(u, \kappa'(u, \dot{x}), \dot{x}) \urcorner) \leftrightarrow \text{Prov}_{T^+}(\ulcorner \Gamma \phi(\dot{x}) \urcorner) \quad (29)$$

$$\leftrightarrow \text{Prov}_T(\ulcorner \Gamma \phi(\dot{x}) \urcorner). \quad (30)$$

Now we are in a position to invoke Theorem 3. Since T^+ is a finite and $\Pi_2^{\bar{\kappa}, \bar{v}}$ axiomatized extension of $EA^{\bar{\kappa}, \bar{v}}$, we have

$$[T^+, \Sigma_1^{\bar{\kappa}, \bar{v}}\text{-IR}] \vdash \forall x (\text{Prov}_{T^+}(\ulcorner \exists u \phi_0(u, \kappa'(u, \dot{x}), \dot{x}) \urcorner) \rightarrow \exists u \phi_0(u, \kappa'(u, x), x)),$$

and then (30) yields

$$[T^+, \Sigma_1^{\bar{\kappa}, \bar{v}}\text{-IR}] \vdash \forall x (\text{Prov}_T(\ulcorner \Gamma \phi(\dot{x}) \urcorner) \rightarrow \phi(x)).$$

So, we can find $\Sigma_1^{\bar{\kappa}, \bar{v}}$ formulas $I_1(x), \dots, I_k(x)$ such that, for each i ,

$$T^+ \vdash I_i(0) \wedge \forall x (I_i(x) \rightarrow I_i(x+1)) \quad (31)$$

and

$$T^+ + \{\forall x I_i(x) \mid i = 1, \dots, k\} \vdash \forall x (\text{Prov}_T(\ulcorner \Gamma \phi(\dot{x}) \urcorner) \rightarrow \phi(x)). \quad (32)$$

Since $(\cdot)^-$, being a non-relativizing interpretation, distributes over boolean connectives and quantifiers, from (31) we obtain Σ_2 formulas $I_1^-(x), \dots, I_k^-(x)$ such that

$$T \vdash I_i^-(0) \wedge \forall x (I_i^-(x) \rightarrow I_i^-(x+1))$$

for all i . And (32) implies that

$$T + \{\forall x I_i^-(x) \mid i = 1, \dots, k\} \vdash \forall x (\text{Prov}_T(\ulcorner \Gamma \phi(\dot{x}) \urcorner) \rightarrow \phi(x)),$$

so we obtain

$$[T, \Sigma_2\text{-IR}] \vdash \forall x (\text{Prov}_T(\ulcorner \Gamma \phi(\dot{x}) \urcorner) \rightarrow \phi(x)).$$

This completes the proof of the main part of Theorem 4 for $n=1$. The other part is no different from that of Theorem 2.

Now we sketch a proof of Theorem 4 for an arbitrary $n > 1$. We consider the case of even n (the case of odd n is only notationally different). Our proof generalizes the one given for $n=1$ fairly straightforwardly, the only problem is not to get confused by various indices of formulas, functions, and variables.

Let T be an extension of IS_{2n} with the only non- EA Π_{2n+2} axiom

$$\tau := \forall u_0 \exists v_0 \forall u_1 \exists v_1 \cdots \forall u_n \exists v_n \tau_0(u_0, v_0, \dots, u_n, v_n),$$

where τ_0 is bounded, and let

$$\phi(x) := \exists v_0 \forall u_1 \exists v_1 \cdots \forall u_n \exists v_n \phi_0(x, v_0, \dots, u_n, v_n)$$

be an arbitrary Σ_{2n+1} formula, with ϕ_0 bounded. As before, we have to show that

$$[T, \Sigma_{2n+1}\text{-IR}] \vdash \forall x (\text{Prov}_T(\ulcorner \phi(\dot{x}) \urcorner) \rightarrow \phi(x)).$$

Let us denote, for $0 \leq k < n$,

$$\begin{aligned} \tau_{2k+1}(u_0, v_0, \dots, u_{n-k}) &:= \exists v_{n-k} \forall u_{n-k+1} \dots \exists v_n \tau_0 \\ \tau_{2k+2}(u_0, v_0, \dots, v_{n-k-1}) &:= \forall u_{n-k} \exists v_{n-k} \dots \exists v_n \tau_0 \\ \phi_{2k+1}(x, v_0, \dots, u_{n-k}) &:= \exists v_{n-k} \forall u_{n-k+1} \dots \exists v_n \phi_0 \\ \phi_{2k+2}(x, v_0, \dots, v_{n-k-1}) &:= \forall u_{n-k} \exists v_{n-k} \dots \exists v_n \phi_0 \end{aligned}$$

Obviously, $\tau_{2k+1}, \phi_{2k+1} \in \Sigma_{2k+1}$ and $\tau_{2k+2}, \phi_{2k+2} \in \Pi_{2k+2}$. Next we introduce new unary function symbols v_1, \dots, v_{2n} and $\kappa_1, \dots, \kappa_{2n}$ and the following formulas:

$$\begin{aligned} \psi_{2k+1} &:= \forall x \forall z \leq x [\exists v_{n-k} \tau_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k}) \\ &\quad \rightarrow \exists v_{n-k} \leq v_{2k+1}(x) \tau_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k})] \\ \psi_{2k+2} &:= \forall x \forall z \leq x [\exists u_{n-k} \neg \tau_{2k+1}((z)_0, \dots, (z)_{2(n-k)-1}, u_{n-k}) \\ &\quad \rightarrow \exists u_{n-k} \leq v_{2k+2}(x) \neg \tau_{2k+1}((z)_0, \dots, (z)_{2(n-k)-1}, u_{n-k})] \\ \theta_{2k+1} &:= \forall x \forall z \leq x [\exists v_{n-k} \phi_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k}) \\ &\quad \rightarrow \exists v_{n-k} \leq \kappa_{2k+1}(x) \phi_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k})] \\ \theta_{2k+2} &:= \forall x \forall z \leq x [\exists u_{n-k} \neg \phi_{2k+1}((z)_0, \dots, (z)_{2(n-k)-1}, u_{n-k}) \\ &\quad \rightarrow \exists u_{n-k} \leq \kappa_{2k+2}(x) \neg \phi_{2k+1}((z)_0, \dots, (z)_{2(n-k)-1}, u_{n-k})]. \end{aligned}$$

Finally, let the theories U_m , for $m = 1, \dots, 2n$, be obtained from $EA^{v_1, \kappa_1, \dots, v_m, \kappa_m}$ by adding the axioms $\psi_1, \theta_1, \dots, \psi_m, \theta_m$ together with the monotonicity axioms for all Skolem functions in the language of U_m :

$$\begin{aligned} x \leq y &\rightarrow v_i(x) \leq v_i(y), \quad 1 \leq i \leq m, \\ x \leq y &\rightarrow \kappa_i(x) \leq \kappa_i(y), \quad 1 \leq i \leq m. \end{aligned}$$

Lemma 9.6. *There is a non-relativizing interpretation $(\cdot)^-$ of U_{2n} in $I\Sigma_{2n}$ such that, for each $1 \leq m \leq 2n$,*

- (a) $(\cdot)^-$ interprets U_m in $I\Sigma_m$;
- (b) $(\cdot)^-$ is identical on formulas in the language of EA ;
- (c) If $A \in \Sigma_1^{v_1, \kappa_1, \dots, v_m, \kappa_m}$ then $(A)^-$ is equivalent to a Σ_{m+1} formula in $I\Sigma_m$.

Proof. Essentially the same proof as for Lemma 9.4. For example, $v_{2k+1}(x)$ is interpreted as the least y such that

$$\begin{aligned} \forall z \leq x [\exists v_{n-k} \tau_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k}) \\ \rightarrow \exists v_{n-k} \leq y \tau_{2k}((z)_0, \dots, (z)_{2(n-k)}, v_{n-k})], \end{aligned}$$

and similarly for the other functions. Totality of v_i and κ_i , for $i \leq m$, together with the axioms ψ_i and θ_i then follows from the strong Σ_i collection schema, which is available in IS_m for $i \leq m$. Verifying the monotonicity axioms is unproblematic.

The graphs of v_i and κ_i are interpreted as $\Delta_0(\Sigma_i)$ formulas. By Proposition 5.11 $\Delta_0(v_1, \kappa_1, \dots, v_m, \kappa_m)$ induction schema is reducible to the induction schema for formulas elementary in the graphs of all functions v_i and κ_i for $1 \leq i \leq m$, that is, to $\Delta_0(\Sigma_m)$ induction. The latter is contained in IS_m , hence IS_m also interprets $EA^{v_1, \kappa_1, \dots, v_m, \kappa_m}$.

Graphs of $\bar{v}_1, \bar{\kappa}_1, \dots, \bar{v}_m, \bar{\kappa}_m$ are $\Delta_0(\Sigma_m)$, and hence Δ_{m+1} in IS_m . This means that any $\Delta_0(\bar{v}_1, \dots, \bar{\kappa}_m)$ formula, being U_m -equivalent to a Δ_1 formula in the graphs of $\bar{v}_1, \bar{\kappa}_1, \dots, \bar{v}_m, \bar{\kappa}_m$, is Δ_{m+1} in IS_m . This implies property (c) of our interpretation, and property (b) is obvious. \square

Lemma 9.7. *For all $1 \leq m \leq 2n$, the formulas τ_m and ϕ_m are U_m -equivalent to open formulas, provided U_m is formulated in the language with function symbols for $\bar{v}_1, \bar{\kappa}_1, \dots, \bar{v}_m, \bar{\kappa}_m$.*

Proof. By induction on m . Let τ'_{2k} denote an open formula equivalent to τ_{2k} in U_{2k} . We define a function $v'_{2k+1}(u_0, v_0, \dots, u_{n-k})$ as

$$\mu v_{n-k} \leq v_{2k+1}(\langle u_0, v_0, \dots, u_{n-k} \rangle) \cdot \tau'_{2k}(u_0, v_0, \dots, u_{n-k}, v_{n-k}).$$

v'_{2k+1} is elementary in $v_1, \kappa_1, \dots, v_{2k+1}$; hence it can be represented by a term in U_{2k+1} . Besides, since U_{2k+1} contains U_{2k} , we have

$$\begin{aligned} U_{2k+1} \vdash \tau_{2k+1} &\leftrightarrow \exists v_{n-k} \tau'_{2k} \\ &\leftrightarrow \tau'_{2k}(u_0, \dots, u_{n-k}, v'_{2k+1}(u_0, \dots, u_{n-k})). \end{aligned}$$

The latter formula is quantifier-free and will be denoted τ'_{2k+1} . Similarly, we define $v'_{2k+2}(u_0, v_0, \dots, v_{n-k-1})$ as

$$\mu u_{n-k} \leq v_{2k+2}(\langle u_0, v_0, \dots, v_{n-k-1} \rangle) \cdot \neg \tau'_{2k+1}(u_0, v_0, \dots, v_{n-k-1}, u_{n-k}).$$

Then

$$\begin{aligned} U_{2k+2} \vdash \tau_{2k+2} &\leftrightarrow \forall u_{n-k} \tau'_{2k+1} \\ &\leftrightarrow \tau'_{2k+1}(u_0, \dots, v_{n-k-1}, v'_{2k+2}(u_0, \dots, v_{n-k-1})), \end{aligned}$$

as required. The argument for ϕ_k is similar. \square

Corollary 9.8. *For each $1 \leq m \leq 2n$, U_m has a $\Pi_1^{\bar{v}_1, \bar{\kappa}_1, \dots, \bar{v}_m, \bar{\kappa}_m}$ axiomatization.*

Proof. By induction on m . A $\Pi_1^{\bar{v}_1, \bar{\kappa}_1, \dots, \bar{v}_m, \bar{\kappa}_m}$ axiomatization of U_{m+1} is obtained from that of U_m by replacing, in the axioms ψ_{m+1} and θ_{m+1} , the subformulas τ_m and ϕ_m , respectively, by their open counterparts τ'_m and ϕ'_m . \square

Now we define $T^+ := U_{2n} + \tau$ (the language of T^+ is that of U_{2n}). Since T contains IS_n , by Lemma 9.6 T^+ is a provably conservative extension of T . T^+ has a

$\Pi_2^{\tilde{v}_1, \tilde{\kappa}_1, \dots, \tilde{v}_{2n}, \tilde{\kappa}_{2n}}$ axiomatization, and ϕ is provably equivalent to a $\Sigma_1^{\tilde{v}_1, \tilde{\kappa}_1, \dots, \tilde{v}_{2n}, \tilde{\kappa}_{2n}}$ formula within T^+ by Lemma 9.7. This allows us to apply Theorem 3 to T^+ , and to carry through the rest of the proof exactly in the way it was done for the case $n = 1$. \square

10. On $\mathcal{B}(\Sigma_n)$ induction rule

We first analyze the induction rule for boolean combinations of Σ_1 formulas.

Proposition 10.1. $\mathcal{B}(\Sigma_1)\text{-IR} \cong \Sigma_1\text{-IR}$.

Proof. We must show that, for every theory T containing EA ,

$$[T, \mathcal{B}(\Sigma_1)\text{-IR}] \subseteq [T, \Sigma_1\text{-IR}].$$

Suppose $A(x)$ is a $\mathcal{B}(\Sigma_1)$ formula such that T proves

$$A(0) \wedge \forall x (A(x) \rightarrow A(x+1)). \quad (33)$$

We must show that $\forall x A(x)$ is contained in $[T, \Sigma_1\text{-IR}]$. It is easy to see by induction on the complexity of boolean combinations that every $\mathcal{B}(\Sigma_n)$ formula is logically equivalent to both a Σ_{n+1} and a Π_{n+1} formula, that is, is Δ_{n+1} in EA . In particular, $A(x)$ is Δ_2 and (33) is (equivalent to) a Π_2 formula. Let T_0 be the finite subtheory of T axiomatized by (33). By Corollary 7.4 we have $[T_0, \Sigma_1\text{-IR}] \equiv [T_0, \Pi_2\text{-IR}]$, and the latter theory contains $\forall x A(x)$. It follows that $\forall x A(x)$ is provable in $[T, \Sigma_1\text{-IR}]$. \square

Essentially, the same argument works for $\mathcal{B}(\Sigma_n)\text{-IR}$, for arbitrary n , only at the last step we have to apply Corollary 9.1 or 9.2. In this way we obtain the following proposition.

Proposition 10.2. *For $n > 1$ we have*

1. $\mathcal{B}(\Sigma_n)\text{-IR} \leq \Sigma_n\text{-IR}$, i.e., the two rules are interderivable. Moreover, k nested applications of $\mathcal{B}(\Sigma_n)\text{-IR}$ are reducible to $k+1$ nested applications of $\Sigma_n\text{-IR}$.
2. The two rules are, in fact, congruent modulo $I\Sigma_{n-1}$, that is, over theories as strong as $I\Sigma_{n-1}$, k nested applications of $\mathcal{B}(\Sigma_n)\text{-IR}$ are reducible to k nested applications of $\Sigma_n\text{-IR}$.

Open Question: Is $\mathcal{B}(\Sigma_n)\text{-IR}$ congruent to $\Sigma_n\text{-IR}$ for $n > 1$?

11. Conclusion

In this paper we introduced natural notions of reducibility and congruence of rules in formal arithmetic. We classified various forms of induction rules of restricted

arithmetical complexity (over EA) modulo congruence relation. It turned out that these forms, most commonly, fall into one of the three main (distinct) categories: (a) rules congruent to induction axiom *schemata*; (b) rules congruent to Σ_n induction rule Σ_n -IR; (c) rules congruent to Π_n induction rule Π_n -IR.

We gave characterizations of Σ_n -IR and Π_n -IR in terms of iterated reflection principles. These characterizations provide natural axiomatizations for closures of *arbitrary* theories containing EA under these rules. It turns out that the number of iterations of reflection principles precisely corresponds to the depth of nestings of applications of induction rules. This shows, in particular, that the two ways of axiomatizing theories are tightly related.

Besides, these characterizations yield several important corollaries concerning finite (non)axiomatizability of theories axiomatized by induction rules, and give wide sufficient conditions for the equivalence of (closures of theories by) Π_{n+1} and Σ_n induction rules.

Proof-theoretic analysis of provably recursive functions of theories axiomatized by rules allows us to sharpen, and give easy new proofs of, several old results. For example, we prove Peter's theorem on reduction of nested recursion to primitive recursion and Finite Basis Theorem for Kalmar elementary functions. We also reproduce some results of Parsons [11] and Sieg [17, 18], e.g., we show that Parsons' result on Π_2 conservativity of $I\Sigma_1$ over Σ_1 -IR is interderivable with (a particular case of) so-called fine structure theorem on uniform reflection principles of U. Schmerl [14].

I hope the results of this paper will convince the reader of the fact that rules in arithmetic are an interesting independent object of study, and that a detailed analysis how particular rules work not only often reveals peculiar effects, but may have useful applications in other topics of proof theory.

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